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Middle School Students' Understanding of Equal Sign

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Abstract
The present study investigates the understanding of the equals sign among middle school students. The study data were gathered from a total of 433 middle school students (111 in the fifth grade, 106 in the sixth grade, 95 in the seventh grade and 121 in the eighth grade) using a ten-item equals sign test developed by the researcher. The study findings suggest that most students have an operational understanding of the equal sign. The students were most successful in answering true/false number sentences and successful lowest in equivalent equations. It was observed that those who applied a relational strategy to the solving of problems understood the underlying structure of the equation, while those who used an operational strategy were observed to make such errors as "the answer comes next" and "extend the problem".
Keywords: Equal Sign, Middle School Students, Algebra.

Introduction

Algebraic thinking involves the forming of generalizations from experiences with numbers and computations, formalizing these ideas through the use of a meaningful symbol system, and exploring the concepts of pattern and function (Van de Walle, 2007). Kaput (1999) describes five different forms of algebraic reasoning, one of which is the "meaningful use of symbolism". The failure of students in algebra may be attributable to the lack of a good understanding of symbols, which makes the equals sign and the concept of variables stand out (Van de Walle, 2007).

Students need to understand that the equals sign indicates that the quantities on either side of it are equivalent (NCTM, 2000). Understanding the equals sign and its various roles in mathematical equations is critically significant in the development of a broad understanding of arithmetic and algebra (Baroody& Ginsburg, 1983; Blanton et al., 2011; Carpenter et al., 2003; Knuth et al., 2006; MacGregor & Stacey, 1997). Students who have a relational understanding of the equals sign are more likely to correctly solve algebraic equations and simple algebraic word problems (Knuth, et al., 2006).

Focusing only on the operational meaning of the equals sign will have a negative impact on the mathematical success of students in the following grades. If they fail to understand its relational meaning, they may not be able to understand algebraic solution strategies (e.g. adding the same elements to either side of the equation to simplify a statement on one side) (Baroody& Ginsburg, 1982). Those who lack a solid understanding of the equals sign tend to find it hard to solve, interpret and transform equations containing more than one numerological term (Herscovics & Linchevski, 1994).

The mathematical equation principle demands that each side of the equation must have the same value, which makes them interchangeable (Kieran, 1981). Matthews et al. (2012) argue that a mathematical equation is an equivalence relationship, and therefore possesses such properties as symmetry, reflection and transitivity. An equivalence relationship indicated using an equal sign is a basic concept that serves as a key link between arithmetic and algebra (Baroody & Ginsburg, 1983; Carpenter et. al, 2003; Kieran, 1981; Knuth et al., 2006).

Arithmetic problems are usually presented with operations to the left of the equals sign and the answer to the right (e.g. 3 + 4 = 7; McNeil et al., 2006). Students who constantly encounter such problems develop three patterns: First of all, they expect to see the operations on the left side of the equals sign, and a blank on the right side for the answer; secondly, they approach these problems as if all numbers are at the same side of the equals sign; and thirdly, they see the equals sign as an operation symbol, such as "+" and "x" (Baroody & Ginsburg 1983; Carpenter et al., 2003; Kieran 1981; Knuth et al. 2006; McNeil & Alibali, 2005b). Instead of seeing it as a symbol denoting an equivalent relationship, students perceive the equals sign as a stimulant to "do something" (Stephans et al., 2013). When given an open number sentence such as "8 + 4 =? + 5", students may make one of the following three mistakes: (1) Thinking "the answer comes next", they replace "?" with 12; (2) in order to "use all the numbers", they replace "?" with 17; (3) or by "extending the problem", they solve it as 8 + 4 = 12 + 5 = 17 (Carpenter et al., 2003). According to Behr et al. (1976), students with the above mentioned mindset think such equations as 13 = 7 + 6, 6 + 4 = 3 + 7, and 8=8 are incorrect, since there is no operation involved, while considering equations such as 15 = 5 + 10 to be backward operations.

There are two main explanations of how students gain an operational understanding of the equals sign. On the teaching side, constantly encountering such equations as "1 + 1 = □" reinforces the operational meaning, while the other explanation relates to cognitive maturation and development. According to Kieran (1980) and Collis (1974), students can understand the relational meaning of equals sign at around the age of 13 years, which makes this age a transition period from operational to relational meaning. This cognitive maturation argument, however, is objected by Falkner et al. (1999), who demonstrated that even first and second grade students are able to understand the relational meaning of the equal sign. To gain such an understanding, students, under the supervision of their teachers, need to work on questions that are appropriate for their levels, being true/false number sentences, open number sentences (Falkner et al., 1999), and non-operational equations such as 8=8 (Baroody and Ginsburg, 1982). Accordingly, it is understood that students can understand the relational meaning of the equals sign, depending on how mathematics is taught.

Literature contains several studies of the equals sign (Alibali et al., 2007; Knuth et al., 2005; Matthews et al., 2012; Stephans et al., 2013). Stephans et al. (2013) identified three levels of understanding of the equals sign among students: the first and most simple being operational, the second being relational–computational, and the third level being relational–structural. A student who understands the relational–structural meaning of equals sign solves the problem "7 + 3 = ___ + 4" as follows: "4 is one more than 3. Therefore, the number in the blank should be one less than 7, which is 6." A student who understands the relational–computational meaning of the equals sign solves the same problem (7 + 3 = ___ + 4) as follows: "7 plus 3 equals 10. To make the other side of the equation equal 10, the blank is 6." Students who understand the operational meaning, on the other hand, consider the equals sign as a symbol in their calculation of the "answer". Students at this level fill the blank in 5 + 3 = ___ + 3 equation with 11, adding all three numbers together. According to them, the 39 + 121 = 121 + 39 equation is incorrect, since 121 + 39 = 160, which is not equal to 121.

In the study by Knuth et al. (2005) analyzing the understanding of such concepts as equality and variables in middle school students', it was observed that the higher the grade of the student, the more they focused on relational meaning. It was further observed, however, that middle school students mostly prioritize the operational meaning of the equalsign. In the same study, Knuth et al. (2005) argued that understanding the concepts of variables and equality affects the success of students in problem-solving, as well as the strategies and justifications they apply while solving the problems.

In their longitudinal study, Alibali et al. (2007) investigated the development of middle school students in terms of the equals sign and equivalent equations and found the students' understanding of and performance with the equals sign to improve in time. The authors argued that such an improvement in students' understanding of the equals sign boosted their performance in equivalent equations.

The present study investigates the understanding of the equals sign among students in the 5th–8th grades of middle school through an extended assessment involving various questions, such as equals sign definition, open number sentences, true/false number sentences and equivalence equations. The research problem of this study is "How do the understanding of middle-school 5th-8th graders about the equal sign?". The sub-problems of the research are as follows:

1. What is the status of middle-school 5th-8th graders correctly answering the questions about the equality sign?
2. What kind of strategies do middle-school 5th-8thgraders use to solve questions in different equation structures about the sign of equality?
3. What kind of mistakes do middle-school 5th-8thgraders make while solving questions in different equation structures about the equality sign?

Method

In this section, the participants of the research, the data collection tool, the data collection process, the pilot study and the analysis of the data are explained.

Participants

Study data were collected from a total of 433 middle school students studying in the 5th (57 male, 54 female, total 111), 6th (49 male, 57 female, total 106), 7th (47 male, 48 female, total 95) and 8th (57 male, 64 female, total 121) grades. Students study at a middle school in a city center located in eastern Turkey. The fact that the mathematics teachers of the students were open to communication and that the students volunteered to participate in the study were important determinants in the selection of this school. The students were informed about the content of the study, and who volunteered were selected for participation. Participants did not receive any training on the equal sign as part of this study.

Development of the Data Collection Tool

A 10-item Equals Sign Test (EST) developed by the researcher was used as the data collection tool. The test comprises four sections, namely: Equals sign definition; Open number sentences; True/false number sentences; and Equivalence equations. The students were first asked to define the equals sign (Carpenter et al., 2003; Knuth, et al., 2005; Knuth et al., 2006; Knuth, et al., 2008; Matthews, et al., 2012; McNeil et al. 2006; Stephans et. al., 2013), while other question types including open number sentence questions ($6 + 7 = \square + 9$, etc.) (Alibali, 1999; Carpenter et al. 2003; Jacobs et al., 2007; Matthews, et al., 2012; McNeil, 2007; Stephans et. al., 2013); true/false number sentences (Is the following statement true or false? $24 + 13 = 25 + 12$) (Carpenter et al., 2003; Falkner et al., 1999; Matthews, et al., 2012; Rittle-Johnson & Alibali, 1999; Stephans et. al., 2013); and equivalent equations (If $54 + 37 = 91$, please explain whether the following statement is true: $54 + 37 - 12 = 91 - 12$) (Knuth, et al., 2005; Knuth, et al., 2008; Matthews, et al., 2012).

Pilot Study

A pilot study was conducted with 30 students. According to the results of the pilot study, necessary arrangements were made in the question statements. The number of questions was reduced from 12 to 10. The parts that were not understood in the question statements were corrected. The application time of the data collection tool was determined as 40 minutes. In addition, expert opinion was obtained from two mathematics educators. As a result of the pilot study, the number of questions in the data collection tool was determined as 10.

Data Collection

Before applying the EST, students were informed that the answers they give would be used solely for a scientific study and will serve to no other purpose. Thereafter, the EST questions were handed out, and the students were given 40 minutes to answer the questions.

Data Analysis

The study data were analyzed in three stages. In the first stage, the students' answers to the EST questions were coded as "true", "false" and "no answer", which was followed by the second step in which the students' question-solving strategies were determined. In the third step, the mistakes made by the students were identified. After these analyses, a mathematics training researcher coded the answers of 80 (approximately 20 percent of all participants) randomly selected students. The mean inter-rater agreement was calculated as 0.94, and the ratings were discussed until full agreement among the coders was reached. Table 1 presents examples of the correct and incorrect answers.

Table 1 Coding Scheme for Selected Explanation Items

Item	Sample incorrect answers	Sample correct answers
6 + 7 = □ + 9	6 + 7 = 13 + 9	6 + 7 = 4 + 9
7 = 5 + 2 True, False, I Don't Know	No reverse addition True, **False**, I Don't Know	7 = 5 + 2 7 = 7 **True**, False, I Don't Know
Explain whether the statement "73 + 56 = 71 + 58" is correct without adding 73 and 56	73 plus 56 equals 129, not 71.	73 + 56 = 71 + 58 73 minus 2 equals to 71, and 56 plus 2 equals to 58. The statement is correct since the equation was balanced.
If 2 . □ + 9 = 23, then □ = 7. Find what □ refers to in the following equation: 2 . □ + 9 - 5 = 23 - 5.	□ = 23 or □ = 18	2 . □ + 9 - 5 = 23 - 5 2 . □ + 4 = 18 2 . □ = 14 □ = 7

Equals Sign Definition (Item 1)

The students' answers to the first EST question were coded as operational, relational, unclear, other or no answer. "A symbol that reflects the sum of figures" and "the sign that indicates the answer of an operation" were categorized as operational, while such answers as "indicates balance", "indicates that both sides of the equation are the same" were categorized as relational. "Equal" and "it means equality" were categorized as unclear (Stephans et al., 2013; Alibali et al., 2007), and such answers as "used while writing dates" and "used to explain words" were categorized as other. Finally, if no answer was given, the category was no answer (Alibali et all, 2007; Mathews et al. 2012; Stephans et all, 2013).

Open Number Sentences (Items 2a, 2b, 2c)

In the first step of the analysis, the students' answers to these questions were coded as "true (T)", "false (F)" or "no answer (NA)". Regardless of the strategy used, correct answers were coded as "true", and incorrect answers as "false". Questions with no operation or no figure written inside the box were coded as "no answer". In the second step, the strategies used to solve the problems were coded as "operational", "computational relational" or "comparative relational" (Mathews et al. 2012; Stephans et al, 2013). Finally, mistakes were coded as "the answer comes next", "extend the problem", "incorrect comparison" and "other" (Carpenter et al., 2003).

True/false Number Sentences (Items 3a, 3b, 3c)

In the first step of the analysis, the students' answers to the questions were coded as "true (T)", "false (F)" or "no answer (NA)". Regardless of the strategy used, "true" answers were coded as "true", all other options as "false", and no operations or markings as "no answer". In the second step of the analysis, the strategies were coded as operational, computational relational or comparative relational, based on the students' explanations or the operations they used. In the third step of the analysis, the students' incorrect answers were coded as "the answer comes next", "extend the problem", "no reverse addition", "no operator" or "no explanation" (Behr et al., 1976; Carpenter et al., 2003).

Equivalent Equations (Items 4, 5, 6)

In the first step of the analysis, the students' answers to the questions were coded as "true (T)", "false (F)" or "no answer (NA)". In the second step of the analysis (with the exception of question no. 6), the students' problem-solving strategies were coded as operational, computational relational or comparative relational. The strategies used by the students to solve question no. 6, on the other hand, were coded as "recognize equivalence", "solve and compare", "substitution" and "other". In the third step of the analysis, the mistakes made by the students were coded as "the answer comes next", "equation solving mistake" "answer after equals sign", "solving by addition", "no explanation", "false substitution" or "other" (Knuth, et al., 2005; Knuth et al., 2006; Knuth, et al., 2008; Matthews, et al., 2012).

Results

The students' definitions of the equals sign were presented using predetermined codes, while other data garnered from the EST were classified in accordance with the question types (open number sentences, true/false number sentences and equivalent equation). In each section, the students' correct answers were presented as percentages/ frequency; the second step indicated the students' problem solving or explanation strategies; and the third step illustrated the students' mistakes and gave relevant examples.

Equals Sign Definition

The students' definitions of the equals sign were not coded as correct or incorrect. The distribution of their answers or explanations according to predetermined codes is presented in Table 2.

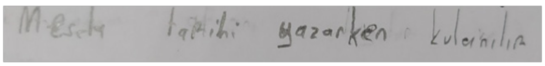
Table 2 Distribution of Students' Equals Sign Definitions

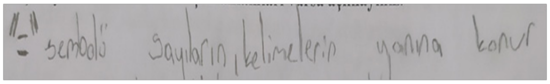
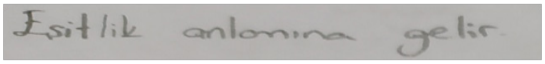
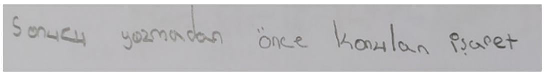
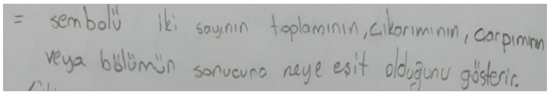
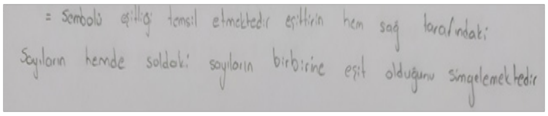
Grade	Operational		Relational		Unclear		Other		No response		Total	
	f	%	f	%	f	%	f	%	f	%	f	%
5.	64	57,5	14	12,6	7	6,3	11	9,9	15	13,5	111	100
6.	81	76,4	15	14,1	3	2,8	1	0,9	6	5,6	106	100
7.	35	36,8	36	37,8	14	14,7	7	7,4	3	3,1	95	100
8.	60	49,5	37	30,5	2	1,7	4	3,3	18	14,8	121	100
Total	240	55,4	102	23,5	26	6	23	5,3	42	9,6	433	100

According to the table above, students from all grades attributed an operational meaning to the equals sign while defining it. The table indicates that 6th (76.4%) and 7th (36.8%) graders used the operational meaning of the equals sign the most and the least, respectively, while the relational meaning was used most by the 7th (37.8%) and the least by the 5th (12.6%) graders. According to Table 2, the students who provided no definition of the equals sign were mostly from the 8th grade. Furthermore, it was observed that the participating students mostly (55.4%) attributed operational meaning to the equals sign, while less than half of the students from each grade provided a relational definition. According to Alibali et al. (2007) and Knuth et al. (2008), this is because the "relational meaning of the equals sign does not develop significantly as grades increase".

The participants' definitions of the equals sign and relevant coding examples are presented in Table 3.

Table 3 Examples of Students' Equals Sign Definitions

Code Type	Sample answers
Other	 (Used when writing history)

Other	 (The equality sign is placed next to the numbers and words)
Unclear	 (It means equality)
Operational	 (The equality sign is written before the result)
Operational	 (The sign of equality indicates the result of adding, subtracting, multiplying and dividing two numbers)
Relational	 (It means the amount on each side is the same)

Open Number Sentences (Items 2a, 2b, 2c)

Data on the correct answers of the students to open number sentences are presented in the table below.

Table 4 Distribution of Students' Answers to Open Number Sentence Questions

Item	Type of response	Grade 5 f (%)	Grade 6 f (%)	Grade 7 f (%)	Grade 8 f (%)
2a	T	35 (31,5)	64 (60,4)	72 (75,8)	108 (89,3)
	F	76 (68,5)	41 (38,7)	20 (21,1)	10 (8,3)
	NA	0 (0)	1 (0,9)	3 (3,2)	3 (2,5)
2b	T	58 (52,3)	75 (70,8)	75 (78,9)	111 (91,7)
	F	47 (42,3)	28 (26,4)	15 (15,8)	5 (4,1)
	NA	6 (5,4)	3 (2,8)	5 (5,3)	5 (4,1)
2c	T	52 (46,8)	73 (68,9)	76 (80)	109 (90,1)
	F	57 (51,4)	32 (30,2)	16 (16,8)	7 (5,8)
	NA	2 (1,8)	1 (0,9)	3 (3,2)	5 (4,1)

According to Table 4, the 5th, 6th and 8th grade students were most and least successful at questions 2b (7 + 5 = 6 + □) and 2a (6 + 7 = □ + 9), respectively, while the 7th grade students were the most and least successful at questions 2c (8 + □ = 9 + 5) and 2a (6 + 7 = □ + 9), respectively, indicating that they were least successful at the same question. The success of students at open number sentence questions was observed to decrease when the equation digressed from the "a + b = c" format. This can be attributed to students' equals sign experience, which is mostly limited to the operations - equals to - answer format (a + b = c) (Li et al., 2008; McNeil & Alibali, 2005b).

The strategies used to answer the questions in this section are presented in the table below. In Table 5, no strategy was coded for unanswered questions.

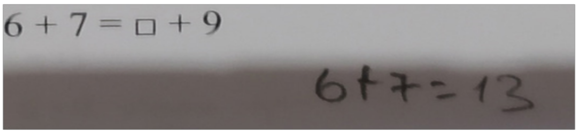
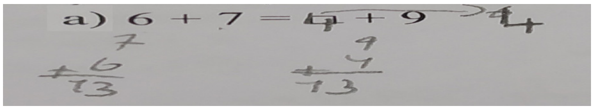
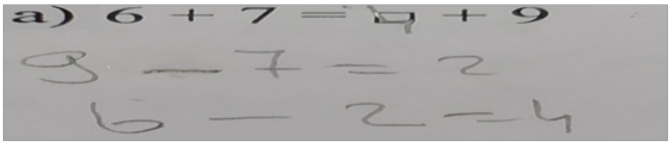
Table 5 Strategies used to Solve Open Number Sentence Questions

Item	Strategy code	Grade 5 f (%)	Grade 6 f (%)	Grade 7 f (%)	Grade 8 f (%)
2a	Operational	76 (68,5)	41 (38,7)	20 (21,1)	10 (8,3)
	Computational relational	35 (31,5)	64 (60,4)	71 (74,7)	108 (89,3)
	Comparative relational	-	-	1 (1,1)	-
2b	Operational	47 (42,3)	28 (26,4)	15 (15,8)	5 (4,1)
	Computational relational	58 (52,3)	75 (70,8)	75 (78,9)	111 (91,7)
	Comparative relational	-	-	-	-
2c	Operational	57 (51,4)	32 (30,2)	16 (16,8)	7 (5,8)
	Computational relational	52 (46,8)	73 (68,9)	76 (80)	109 (90,1)
	Comparative relational	-	-	-	-

According to Table 5, 6th, 7th and 8th grade students mostly adopted a computational relational strategy in all questions, while 5th grade students used an operational strategy for questions 2a and 2c, and opted for a computational relational strategy for question 2b. One of the most striking findings of the table is that only one student used a comparative relational strategy, which was not used in any other question.

Examples of the strategies used to solve the open number sentence questions are presented in the table below.

Table 6 Examples of Strategies used to Solve Open Number Sentence Questions

Strategy code	Sample answers
Operational	
Computational relational	
Comparative relational	

The distribution of the mistakes made while solving open number sentence questions are presented in the table below.

Table 7 Mistakes made in the Open Number Sentence Questions

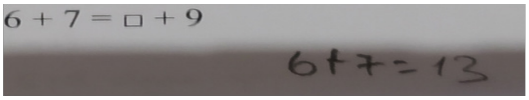
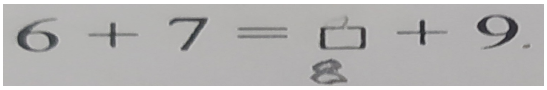
Item	Type of mistake	Grade 5 (f)	Grade 6 (f)	Grade 7 (f)	Grade 8 (f)
2a	The answer comes next	35	23	14	6
	Extend the problem	36	18	6	3
	Incorrect comparison	5	-	-	1
2b	The answer comes next	14	16	8	1
	Extend the problem	27	12	6	2
	Incorrect comparison	6	-	1	2

2c	The answer comes next	31	26	15	5
	Extend the problem	23	6	1	1
	Incorrect comparison	3	-	-	

According to Table 7, 5th, 6th and 7th grade students mostly made "the answer comes next" mistakes in all three questions; while 8th grade students made "the answer comes next" mistake more than the other mistakes in questions 2a (6 + 7 = □ + 9) and 2c (8 + □ = 9 + 5). The second most common mistake was "extend the problem". Furthermore, according to the data presented in Table 7, the greatest number of mistakes were made in question 2a, while the least number of mistakes were made in question 2b (7 + 5 = 6 + □).

Examples of the mistakes made in the open number sentence questions are presented in the table below.

Table 8 Examples of the Mistakes Made in Open Number Sentence Questions

<table>
<tr><th>Type of mistake</th><th>Sample answers</th></tr>
<tr><td>Extend the problem</td><td>
</td></tr>
<tr><td>The answer comes next</td><td>
</td></tr>
<tr><td>Incorrect comparison</td><td>
</td></tr>
<tr><td>Incorrect comparison</td><td>
</td></tr>
</table>

The mistakes "extend the problem" and "the answer comes next" presented in Table 8 are commonly mentioned in literature (Carpenter et al., 2003; Falkner et al., 1999; Matthews, et al., 2012; Stephans et all, 2013), and are thought to be associated with the operational meaning attributed to the equals sign. Furthermore, the frequent encounters of students with equations in the "a + b = c" format can be considered another cause of such mistakes. In "incorrect comparison" mistakes, on the other hand, the students failed to realize that when one of the terms increased (or decreased), the other also had to decrease (or increase) proportionally.

True/False Number Sentences (Items 3a, 3b, 3c)

Data on the students' correct answers to true/false number sentence questions are presented in the table below.

Table 9 Distribution of Students' Answers to True/False Number Sentence Questions

Item	Type of response	Grade 5 f (%)	Grade 6 f (%)	Grade 7 f (%)	Grade 8 f (%)
3a	T	89 (80,2)	91 (85,8)	85 (89,5)	112 (92,6)
	F	8 (7,2)	4 (3,8)	4 (4,2)	1 (0,8)
	NA	14 (12,6)	11 (10,4)	6 (6,3)	8 (6,6)
3b	T	93 (83,8)	99 (93,4)	88 (92,6)	120 (99,2)
	F	10 (9)	2 (1,9)	2 (2,1)	0 (0)
	NA	8 (7,2)	5 (4,7)	5 (5,3)	1 (0, 8)

3c	T	74 (66,7)	82 (77,4)	82 (86,3)	116 (95,9)
	F	24 (21,6)	16 (15,1)	5 (5,3)	2 (1,7)
	NA	13 (11,7)	8 (7,5)	8 (8,4)	3 (2,5)

Table 9 indicates that the students were most successful at solving question 3b (7 = 5 + 2), regardless of their grade. Accordingly, 5th, 6th and 7th grade students were least successful at question 3c (24 + 13 = 25 + 12), while 8th grade students mostly failed at question 3a (4 = 4). Furthermore, it can be understood from Tables 5, 9 and 14 that the students achieved the highest success rates in the true/false number sentence questions, concurring with the results of Stephans et al. (2013).

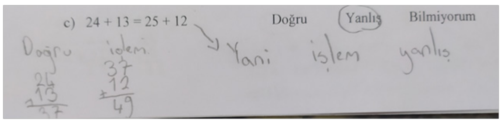
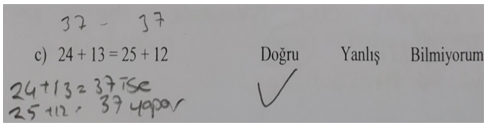
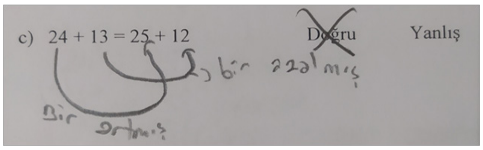
The strategies used to correctly answer the questions in this section are presented in the table below. No strategy was coded for unanswered questions.

Table 10 Strategies used to Solve True/False Number Sentence Questions

Item	Strategy code	Grade 5 f (%)	Grade 6 f (%)	Grade 7 f (%)	Grade 8 f (%)
3a	Operational	8 (7.2)	4 (3.8)	4 (4.2)	1 (0.8)
	Computational relational	89 (80.2)	91 (85.8)	85 (89.5)	112 (92.6)
	Comparative relational	-	-	-	-
3b	Operational	10 (9)	2 (1.9)	2 (2.1)	-
	Computational relational	93 (83.8)	99 (93.4)	88 (92.6)	120 (99.2)
	Comparative relational	-	--	-	-
3c	Operational	24 (21.6)	16 (15.1)	5 (5.3)	2 (1.7)
	Computational relational	72 (65)	76 (71.7)	80 (84.2)	112 (92.6)
	Comparative relational	2 (1.8)	6 (5.7)	2 (2.1)	4 (3.3)

According to Table 10, the students used the "computational relational" strategy more than the other strategies to solve true/false number sentence questions. It was further observed that a "comparative relational" strategy was used only in question 3c. Examples of the strategies used by students are presented in Table 11.

Table 11 Examples of Strategies used to Solve True/False Number Sentences Questions

Strategy code	Sample answers
Operational	
Computational relational	
Comparative relational	 (Bir artmış: 1 increased, Bir azalmış: 1 decreased)

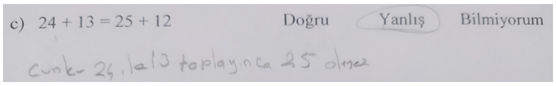
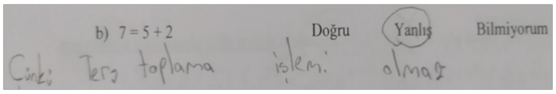
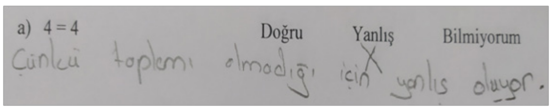
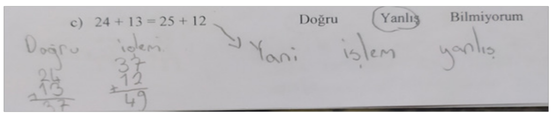
The distribution of the mistakes made while solving true/false number sentence questions is presented in the table below.

Table 12 Students' Mistakes in True/False Sentence Questions

Item	Type of mistake	Grade 5	Grade 6	Grade 7	Grade 8
3a	No explanation	6	1	3	-
	The answer comes next	-	-	1	-
	No operator	3	4	1	1
3b	No explanation	3	-	1	-
	The answer comes next	3	-	1	-
	No reverse addition	4	2	-	-
3c	No explanation	11	6	1	2
	The answer comes next	5	4	2	-
	Extend the problem	8	6	2	-

Table 12 indicates that the number of mistakes varies depending on the student grade and question type. For example, 5th grade students mostly made "no explanation" mistakes in question 3a, while the most common mistake among 6th graders was "no operator" to the same question. Furthermore, "no explanation", "no operator" and "no reverse addition" mistakes observed in true/false sentence questions did not stand out in the open number sentence questions, which suggests that such mistakes are attributable to the question types. Similar mistake types were encountered in the study by Matthews et al. (2012), and such mistakes are thought to depend on the question types. Examples of the mistakes made by the students are presented in Table 13.

Table 13 Examples of Mistakes Made by Students in True/False Number Sentence Questions

Type of Mistake	Sample Answers
The answer comes next	 (Adding 24 to 13 doesn't make 25)
No reverse addition	 (Because, there is no reverse addition)
Only answer or No explanation	
No operator	 (Incorrect as there is no addition)
Extend the problem	

Equivalent Equations (Items 4, 5 And 6)

Data on the students' correct answers to equivalent equation questions are presented in the table below.

Table 14 Distribution of Students' Answers to Equivalent Equation Questions

Item	Type of response	Grade 5 f (%)	Grade 6 f (%)	Grade 7 f (%)	Grade 8 f (%)
4	T	61 (55)	80 (75,5)	80 (84,2)	110 (90,9)
	F	36 (32,4)	13 (12,3)	8 (8,4)	3 (2,5)
	NA	14 (12,6)	13 (12,3)	7 (7,4)	8 (6,6)
5	T	21 (18,9)	35 (33)	41 (43,2)	61 (50,4)
	F	82 (73,9)	61 (57,5)	48 (50,5)	49 (40,5)
	NA	8 (7,2)	10 (9,4)	6 (6,3)	11 (9,1)
6	T	52 (46,8)	68 (64,2)	77 (81,1)	95 (78,5)
	F	33 (29,7)	20 (18,9)	5 (5,3)	11 (9,1)
	NA	26 (23,4)	18 (17)	13 (13,7)	15 (12,4)

According to Table 14, the students were most and least successful at questions 4 (If 54 + 37 = 91, please explain whether the following statement is true: 54 + 37 - 12 = 91 - 12) and 5 (Explain whether the statement "73 + 56 = 71 + 58" is correct without adding 73 and 56), respectively. Among the questions in the data collection tool, the students were least successful at question 5. With the exception of 8th graders, less than 50 percent of the students were able to answer this question correctly, which is attributed to the question structure, which required the use of a comparative relational strategy to find the correct answer. Tables 2, 5 and 10 indicate that the least popular strategy among the students was the comparative relational strategy, which is so unpopular that no student used it for questions 2b or 2c, as presented in Table 5.

According to Table 14, the higher the grade, the better the rates of correct answers to questions 4 and 5. In question 6 (If 2 . □ + 9 = 23, then □ = 7 Find what □ refers to in the following equation: 2 . □ + 9 - 5 = 23 - 5), on the other hand, 7th grade students were more successful than 8th graders.

The strategies used to answer equivalent equation questions are presented in the table below.

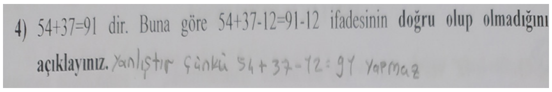
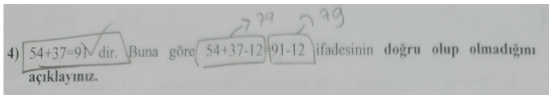
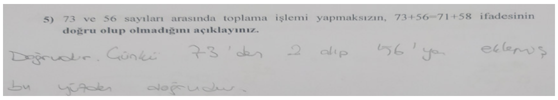
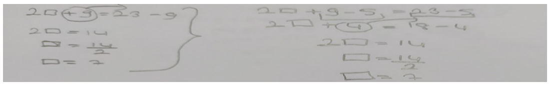
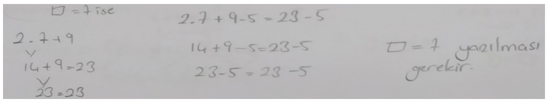
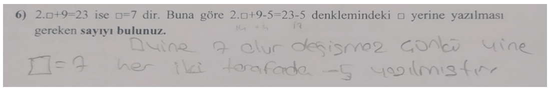
Table 15 Strategies Used to Solve Equivalent Equation Questions

Item	Strategy code	Grade 5 f (%)	Grade 6 f (%)	Grade 7 f (%)	Grade 8 f (%)
4	Operational	36 (32,4)	13 (12,3)	8 (8,4)	3 (2,5)
	Computational relational	54 (48,6)	70 (66)	63 (66,3)	77 (63,6)
	Comparative relational	7 (6,3)	10 (9,4)	17 (17,8)	33 (27,2)
5	Operational	20 (18)	15 (14,1)	11 (11,5)	13 (10,7)
	Computational relational	62 (55,8)	46 (43,3)	37 (38,9)	36 (29,7)
	Comparative relational	21 (18,9)	35 (33)	41 (43,2)	61 (50,4)
6	Substitution	73 (65,7)	58 (54,7)	31 (32,6)	15 (12,3)
	Solve and compare	5 (4,5)	15 (14,1)	44 (46,3)	76 (62,8)
	Recognize equivalence	1 (0,9)	1 (0,9)	-	7 (5,7)
	Answer after equal sign	4 (3,6)	4 (3,7)	1 (1,0)	4 (3,3)
	Other	2 (1,8)	10 (9,4)	6 (6,3)	4 (3,3)

According to the results of question 4, as presented in Table 15, the use of the operational strategy decreases while computational relational and comparative relational strategies gain popularity at higher grades. The results of question 5 indicate that operational strategy loses popularity, as opposed to the comparative relational strategy, which gains prevalence with increasing grades. Accordingly, students were able to use the comparative relational strategy properly in question 5 at a rate of between 18 and 50 percent. For question 6, the most popular strategy among 5th and 6th graders was substitution, while 7th and 8th graders used the solve and compare strategy the most. Having said that, the recognize equivalence strategy was not used by 7th grade students, which is one of the striking findings presented in Table 15.

Examples of the strategies used to solve equivalent equation questions are presented in the table below.

Table 16 Examples of the Strategies used to Solve Equivalent Equation Questions

Strategy code	Sample answers
Operational	 4) 54+37=91 dir. Buna göre 54+37-12=91-12 ifadesinin doğru olup olmadığını açıklayınız. Yanlıştır çünkü 54+37-12=91 yapmaz (False. Because 54+37-12≠91)
Computational relational	 4) 54+37=91 dir. Buna göre 54+37-12=91-12 ifadesinin doğru olup olmadığını açıklayınız.
Comparative relational	 5) 73 ve 56 sayıları arasında toplama işlemi yapmaksızın, 73+56=71+58 ifadesinin doğru olup olmadığını açıklayınız. (73 minus 2 equalsto 71, and 56 plus 2 equalsto 58)
Solve and compare	
Substitution	
Recognize equivalence	 6) 2.□+9=23 ise □=7 dir. Buna göre 2.□+9-5=23-5 denkleminde □ yerine yazılması gereken sayıyı bulunuz. (The answer is 7. Because 5 is subtracted from both sides.)

The distribution of the mistakes made in the equivalent equation questions is presented in the table below.

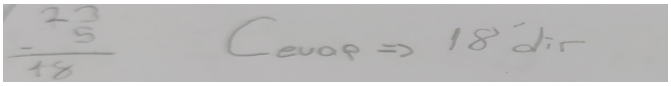
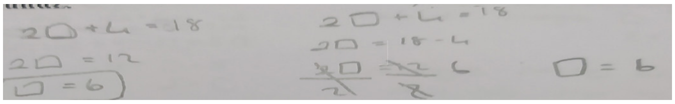
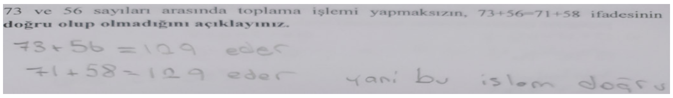
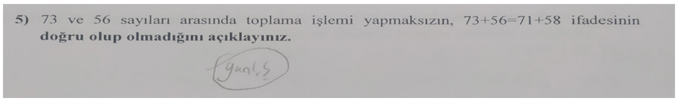
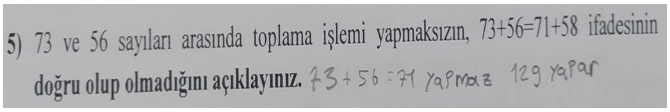
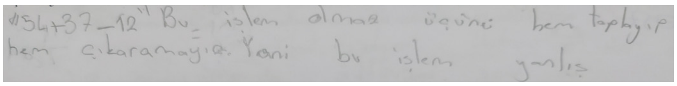
Table 17 Examples of the Mistakes Made in Equivalent Equation Questions

Item	Type of mistake	Grade 5	Grade 6	Grade 7	Grade 8
4	The answer comes next	33	11	7	2
	Other	3	1	-	-
	Only answer or No explanation	-	1	1	1
5	Solving by addition	55	46	32	44
	The answer comes next	15	11	6	1
	Other	2	3	6	3
6	Only answer or no explanation	10	1	3	1
	False substitution	25	15	4	7
	Equation solving mistake	2	1	-	-
	Answer after equal sign	4	4	1	4
	Other	2	-	-	-

According to the results presented in Table 17, the students mostly made “the answer comes next” mistake in question 4, “solving by addition” in question 5, and “false substitution” in question 6.

"The answer comes next" mistakes are attributable to the operational symbol meaning attributed to the equals sign. Those who made the "solving by addition" mistake, on the other hand, were the same students who used the computational relational strategy. Even though their strategy, operation and explanation were correct, their answers were deemed incorrect due to the "without adding …" statement used in the question. Additionally, "equation solving mistakes" were attributed to their lack of knowledge of arithmetical operations. According to Table 18, the student who made that mistake wrote "18 - 4 = 12", and gave an incorrect answer due to an arithmetical error.

Table 18 Examples of the Mistakes Made in Equivalent Equation Questions

Type of mistake	Sample Answer
Answer after equal sign	
Equation solving mistake	
Solving by addition	 73 ve 56 sayıları arasında toplama işlemi yapmaksızın, 73+56=71+58 ifadesinin doğru olup olmadığını açıklayınız. (Yani, işlem doğrudur: That is, operation is true.)
Answer Only	 5) 73 ve 56 sayıları arasında toplama işlemi yapmaksızın, 73+56=71+58 ifadesinin doğru olup olmadığını açıklayınız. (Yanlış: False)
The answer comes next	 5) 73 ve 56 sayıları arasında toplama işlemi yapmaksızın, 73+56=71+58 ifadesinin doğru olup olmadığını açıklayınız. (73 plus 56 equals 129, not 71)
Other	 (This operation is false. Because we cannot do both addition and subtraction.)

Discussion and Conclusion

The present study has sought to clarify middle school students' understanding of the equals sign, making use of a data collection tool that involved asking the students to provide a definition. It was found that 12–37 percent of middle school students gave a relational definition, while 36–76 percent were observed to give an operational definition (see Table 2), indicating that most of the students had an operational view – in other words, they consider the equals sign to be a symbol that "presents a result" or "distinguishes the operations from the result" (Alibali, 1999; Baroody & Ginsburg, 1983; Falkner et al., 1999; Li et al., 2008; Matthews et al., 2012; McNeil, 2007; Stephans et al., 2013). Even though most of the students had an operational view of equals sign, they found it hard to understand the underlying structure of arithmetical equations (Stephens et al., 2013).

The data collection tool also included open number sentence questions aimed at determining the students' understanding of the equal sign. It was observed that the higher the grade, the more they used the computational relational strategy to solve problems and to reach correct answers (see Tables 4 and 5). It was also observed that students focused more on relational meaning of equals sign in this question than they did in the question asking for a definition. It can be argued that most 6th, 7th, and 8th grade students were able to understand the underlying structure of the equation, unlike in the 5th graders, who mostly came up with incorrect answers when using an operational strategy.

The true/false number sentence questions in the data collection tool were aimed at determining the students' understanding of the equal sign. The students were most successful in this question type when compared to the other question types used in the data collection tool (see Table 9). Based on the students' explanations of their answers, it can be understood that they focused specifically on the computational relational meaning of the equals sign, and approximately 3 percent of the students used a comparative relational strategy to answer question 3c (24 + 13 = 25 + 12, True, False, I Don't Know) (see Table 10). Literature suggests using these types of questions to encourage students to use the relational meaning of the equals sign (Carpenter et all, 2003; Matthews et all, 2012; Stephans et all, 2013), and the findings of the present study support this.

Equivalent equation questions were also included in the data collection tool, aimed at determining the students' understanding of the equal sign. This group of questions included question 5 (Explain whether the statement "73 + 56 = 71 + 58" is correct without adding 73 and 56.), which recorded the lowest correct answer rate. The students used a comparative relational strategy most for this question and performed the worst (see Tables 14 and 15). Those who used computational relational and comparative relational strategies in question 4, and a comparative relational strategy in question 5, and that recognized the equivalence strategy in question 6, can be argued to have understood the underlying structure of equations. It cannot be ascertained whether those who came up with correct answers using other strategies were able to recognize such underlying structures. To reach such a conclusion, students should be interviewed, and detailed research should be carried out. The above-mentioned findings agree with similar studies in the literature (Alibali et al., 2007; Knuth et al., 2005; Matthews et al., 2012; Stephans et al., 2013).

Recommendations

To improve the understanding of students of the equals sign, true/false number sentence, open number sentence and equivalent equation questions should be used, respectively, in their education. Textbooks and workbooks should contain such questions, and they should be presented to students during mathematics classes.

The popularity of the operational meaning of the equals sign among students can be attributed to their experience, which is limited to standard arithmetical problems in the operation - equals to - answer (a + b =?) format (Li et al., 2008). However, this experience leads to such beliefs that "an equals sign is followed by an answer", which does not reflect the essence of equality in mathematics, and such strategies as "extend the problem". In order to improve the understanding of equals sign among students, such questions as "a + b =? + c", a + b = c +?" and "a +? = b + c" should be used, as it is thought such questions will help students leave the operational meaning of the equals sign behindand learn its relational meaning.

To encourage students to adopt a comparative relational strategy, such questions as "Explain whether the statement "73 + 56 = 71 + 58" is correct without adding 73 and 56" should be used.

Students should also be encouraged to understand the underlying structure of equations, instead of solely finding the correct answer in questions that involve equals signs.

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