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Mathematical Models of Single Stage Monod Model of Landfill Degradation using Laplace Transform and Homotopy Perturbation Method

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Abstract

A mathematical model to predict waste degradation and landfill gas production is discussed. The model is based on two reaction-diffusion equations containing a non-linear term related to Michaelis–Menten kinetics of the enzymatic reaction. In this paper, we present an approximate analytical solution of the non-linear differential equations that describe the diffusion coupled with a Michaelis–Menten kinetics law. Approximate analytical expressions for substrate and biomass concentrations have been derived for all values of parameter using Laplace transform method and homotopy analysis method. These results are compared with the numerical result and satisfactory is noted. The obtained results are valid for the whole solution domain.

Keywords: Mathematical modeling, Non-linear differential equations, Laplace Transform method, Homotopy perturbation method, Landfill modeling, Michaelis–Menten kinetics.

Introduction

Landfill gas is a complex mix of different gases created by the action of microorganisms within a landfill. A landfill gas generation model is a tool that simulates in simple terms the complex changes that occur during decomposition of waste in a landfill (Lamborn 2012). Mathematical modeling of the biochemical, physical and chemical processes inside landfills has been widely reported (Young, 1989; El Fadel et al., 1996; Haarstrick et al., 2001; White, 2004). Most

models focus on the description of these processes in the anaerobic period, when the biodegradable organic waste is transformed into biogas. Modelling of moisture-dependent aerobic degradation of solid waste is discussed Lefebvre (2008). Bestamin Ozkaya et al. (2007) studied the neural network prediction model for the methane fraction in biogas from field-scale landfill bioreactors. Kirthiga (2020), The single-stage and multi-stage Monod landfill degradation model is based on a coupled system of rate equations containing a nonlinear term related to Michaelis– Menten kinetics of the enzymatic reaction solved analytical solution using a new approach of the homotopy perturbation method.

Recently Lamborn (2012) derive the mathematical equation for a single-stage Monod model for landfill gas generation. To my knowledge, no rigorous analytical solutions of non-linear equation in biomass and solid concentration in single-stage Monod model under non-steady-state conditions for all values or reaction/diffusion parameters have been reported. The purpose of this communication is to derive approximate analytical expressions for the non-steady-state concentrations of biomass and substrate for all values of parameters using the new approach of homotopy perturbation method.

Mathematical Formulation

In examining the accuracy of model predictions and looking at the effects of scale, it is worthwhile having several models to compare. This single-stage Monod model has the following form and this model is based on the work by Monod (1942), El-Fadel et al. (1996) and White et al. (2004). The non-linear differential equations presented below represent the rate of change of biomass and substrate for a single-stage model.

$$\frac{d\,C_b}{d\,t} = \left[\left(\frac{\grave{\imath}_b\,C_s}{K_b + C_s}\right) - K_{d\,b}\right] C_b \tag{1}$$

$$\frac{d\,C_s}{d\,t} = -\frac{1}{Y_s}\frac{\grave{\imath}_b C_s}{K_b + C_s} C_b \tag{2}$$

where C_b is the biomass carbon concentration *(kg/m^3)*, C_s is the substratecarbon concentration *(kg/m^3)*, K_b the half saturation constant for biomass *(kg/m^3)*, and K_b is the biomass death rate constant (1/day) respectively. Y_s is the mass of biomass formed per mass of substrate carbon utilized *(kg/kg)* and μ_b is the mass specific growth rate constant(1/day).The initial conditions are

$$C_b = C_{bi},\ C_s = C_{si} \text{ at } t = 0 \tag{3}$$

Method of Solution: Homotopy Perturbation Method and Laplace Transform Method

Consider the function

$$A(u) - f(r) = 0 \tag{4}$$

with the boundary condition of

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0 \tag{5}$$

where A(u) is defined as

$$A(u) = L(u) - N(u) \tag{6}$$

Homotopy Perturbation procedure is shown as:

$H(v,p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0$ (7)

or

$H(v,p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0$ (8)

According to the homotopy perturbation method form of the Eqns. (1) and (2) is constructed as follows

$$H(\bar{u}, p) = (1-p)\left(\frac{dC_b}{dt} + K_{db}C_b\right) + p\left(\frac{dC_b}{dt} - \frac{\mu_b C_s}{K_b + C_{s_i}}C_b + K_{db}C_b\right) = 0 \quad (9)$$

$$H(\bar{\theta}, p) = (1-p)\left(\frac{dC_s}{dt} + \frac{\mu_b C_s}{Y_s(K + C_{s_i})}C_b\right) + p\left(\frac{dC_s}{dt} + \frac{\mu_b C_s}{Y_s(K_b + C_{s_i})}C_b\right) = 0 \quad (10)$$

The approximate solution of the equations (9) and (10) are

$C_b = C_{b0}p^0 + C_{b1}p^1 + C_{b2}p^2 + ...$ (11)

$C_s = C_{s0}p + C_{s1}p^1 + C_{s2}p^2 + ...$ (12)

Substituting Eqs. (11) and (12) into (9) and (10), yields Comparing the coefficients of p, p^1 and solving C_{b0}, C_{b1}, C_{s0} and C_{s1}.

$$p^0 : \frac{dC_{b0}}{dt} + K_{db}C_{b0} = 0 \quad (13)$$

$$p^0 : \frac{dC_{s0}}{dt} + \frac{\mu_b C_{s_i}}{Y_s(K_b + C_{s_i})}C_{b0} = 0 \quad (14)$$

$$p^1 : \frac{dC_{b0}}{dt} + K_{db}C_{b1} - \frac{\mu_b C_s}{K_b + C_{s_i}}C_{b0} = 0 \quad (15)$$

The initial conditions

$t = 0, C_{b0} = C_{b1} = C_{s0} = C_{si}$

$t = 0, C_{b1} = 0, C_{s0} = 0$ (16)

Solving (13) - (15) and (16) Using Laplace transform method as follows:

$$C_{b_0}(s) = \frac{C_b}{s + K_{db}} \quad (17)$$

$$C_{s_0}(s) = \frac{A\left(\frac{C_{si}}{s+K_{db}}\right)+C_b}{\left(s+K_{db}\right)} \quad (18)$$

$$C_{b1}(s) = C_{si} - B\left(\frac{C_{bi}}{s+K_{db}}\right) \quad (19)$$

Taking Inverse Laplace transform

$$C_{b_0}(t) = C_{bi}\, e^{-K_{db}\, t} \quad (20)$$

$$C_{s_0}(t) = \frac{C_{si}\left(C_{si}\, K_{db} + K_{db}\, K_b + C_b\right)}{K_{db}\left(C_{si}+K_b\right)} - \frac{C_{si}\, C_{bi}\, e^{-K_{db}\, t}}{K_{db}\left(C_{si}+K_b\right)} \quad (21)$$

$$C_{b1}(t) = C_{bi}\left(At+1\right)e^{-K_{db}\, t} \quad (22)$$

Adding the coefficients of we get

$$\begin{aligned} C_b(t) &= C_{bo} + C_{b1} \\ &= C_{bi}\, e^{-K_{db}\, t} + C_{bi}\left(At+1\right)e^{-K_{db}\, t} \end{aligned} \quad (23)$$

$$C_{s\,0}(t) = \frac{C_{si}\left(C_{si}\, K_{db} + K_{db}\, K_b + C_{bi}\right)}{K_{db}\left(C_{si}+K_b\right)} - \frac{C_{si}\, C_{bi}\, e^{-K_{db}\, t}}{K_{db}\left(C_{si}+K_b\right)} \quad (24)$$

where,

$$A = \frac{\mu_b C_{si}}{K_b + C_{si}} \text{ and } B = \frac{\mu_b C_{si}}{Y_s(K_b + C_{si})} \quad (25)$$

Numerical Simulation

The non-linear differential equations (1) and (2) are also solved using numerical methods. The function pdex4 in Matlab software which is the function of solving the initial value problems for ordinary differential is used to solve this equation. The Matlab program is also given in Appendix B. The numerical results are also compared with our analytical results in Figs.(2) and (3) . Satisfactory agreement is found for all values of time t.

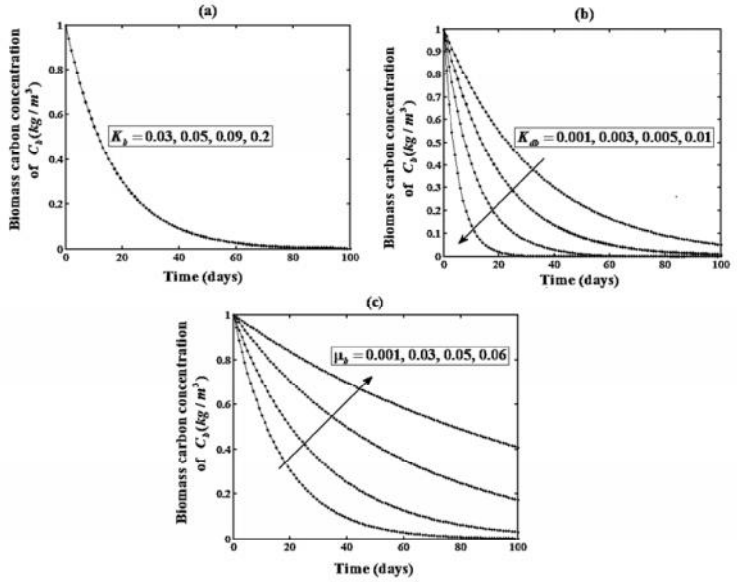

Figure 1: Comparison between Analytical (Eqn. (23)) and Numerical Results for the Concentration of Biomass Carbon Concentration for Various Values of Parameters

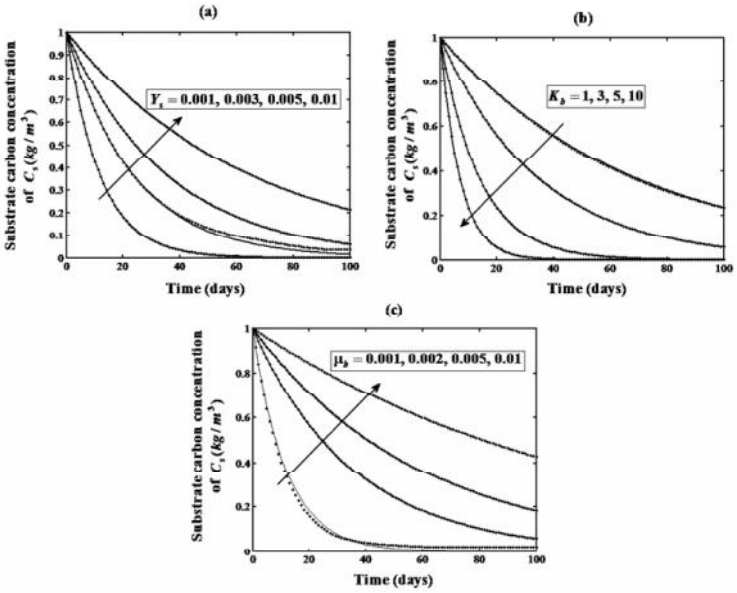

Figure 2: Comparison between Analytical (Eqn. (24)) and Numerical Results for the Concentration of Substrate Carbon Concentration for Various Values of Parameters

Results and Discussion

Equations (4) - (5) are the new analytical expressions of the biomass carbon concentration (C_b) and substrate carbon concentration (C_s) for all values of parameters C_{s1} = 1 kg/m^3, c_{b1} = 1 kg/m^3, μ_b = 0.001 day^{-1}, K_{db} = 0.06 day^{-1}, Y_s = 0.059. Figs. 2 (a)-(c) represents the of biomass

carbon concentration C_b for all values of rate constant. The concentration of biomass carbon was calculated for all small values of the saturation parameter K_b. From the figure, it is observed that the concentration of biomass carbon C_b does not differ significantly for all values of parameters. From the Figs. (2a) and (2c), it is observed that the concentration of biomass carbon decreases when K_{db} (biomass death rate constant) and μ_b (mass-specific growth rate) increases.

Figs.3 (a)-(c) represent the of substrate carbon concentration C_s for all values of rate constant. From the figure, it is observed that the substrate carbon concentration is decreases when half saturation constant for biomass (K_b) . From the Figs. (2a) and (2c),it is inferred that the substrate carbon concentration is increases when mass of substrate concentration (Y_s) and the mass specific growth rate constant (μ_b) increases.

Conclusion

The mathematical model is developed for waste degradation and landfill gas production. A non-linear time-dependent reaction-diffusion equations containing a non-linear term related to Michaelis–Menten kinetics of the enzymatic reaction has been solved analytically using laplace transform method and homotopy perturbation method. The analytical expressions are compared with numerical results using Matlab software. This analytical method is used for other non-linear problems. Good agreement is noted.

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Appendix A: Matlab program of the non-linear equation (1) and (2)

```
function main1
options= odeset('RelTol',1e-6,'Stats','on');
Xo = [1 1];
tspan = [0,10];
tic
[t,X] = ode45(@TestFunction,tspan,Xo,options);
toc
figure
hold on
plot(t, X(:,1))
plot(t, X(:,2))
return
function [dx_dt]= TestFunction(t,x)
mb=.0001;kb=.175;kdb=.06;ys=0.057;
dx_dt(1)=-(((mb*x(2))/(kb+x(2)))-kdb)*x(1);
dx_dt(2)=((mb*x(2)*x(1))/(ys*(kb+x(2))));
dx_dt = dx_dt';
return
```