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A Study on Characteristics of the Graceful Labelling

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The origin of graph theory can be traced back to the work of the Swiss mathematician Leonard Euler (1707-1783) who in 1735 solved a problem that came to be known as the 'Seven Bridges of Konigsberg' [21]. In 1736, Leonard Euler published the first paper on graph theory, where he reported the solution to the above problem. He mathematically proved that it is impossible to find a route traversing all the seven bridges. Graph theory belongs to the well known area in mathematics called 'discrete mathematics'. There exist a lot of fundamental differences between the problems and mode of solutions in the area of discrete mathematics and continuous mathematics.

Discrete mathematics constitutes the number of objects whereas in continuous mathematics their size is measured. Discrete mathematics had evolved as early as man learned counting but continuous mathematics has long dominated the history of mathematics. It began to change in twentieth century. It is seen that the first development occurred in mathematics when it began to reach people. Its core point has varied from the concept of numbers to the concept of set. Set theory is more appropriate to the mode of discrete mathematics than to that of continuous mathematics. The hike in the use of computers was the dramatic point. As graph theory belongs to the area of discrete mathematics, it has got a number of attractive applications in computer science and many other allied streams including engineering and commerce.

Various interesting field of research in graph theory includes the labeling of discrete structures, decomposition of graphs, topological graph theory, algebraic graph theory, fuzzy graph theory and domination in graphs. Graph labeling is a prospective research area due to its vital applications that could challenge our mind for eventual solutions.

Graph Labeling

The concept of labeling of graph was first introduced by Rosa [65] in mid sixties. In labeling, distinct nonnegative integers are associated to the vertices of a graph *G* as vertex labels, so that each edge receives a distinct nonnegative integer as an edge label. If the domain is the vertex set then it is the vertex labeling. If the domain is the edge set then it is called the edge labeling. In total labeling, the domain is both vertexset and edge set.

The field of graph labeling has been creating a lot of interest and motivation among many researchers and this branch of mathematics has found several applications. Labeled graphs have its wide variety of applications in designing communication network addressing systems, determining ambiguities in X-Ray crystallographic analysis, determining the radio astronomy and optimal circuit layouts.

Additionally, graph labeling is also relevant in additive number theory and in coding theory, particularly for missile guidance codes, design of good radar type codes and convolution codes with optimal autocorrelation properties [12, 37, 45, 50, 83].

More than two thousand research papers have been published so far in various graph labeling. Some of them have been studied such as graceful labeling, cordial labeling, (n, k)-equitable labeling, E -cordial labeling, totally magic cordial labeling, multiplicative labeling, multiplicative divisor labeling and stronglymultiplicative labeling.

The ongoing research in graph labeling is reorganized and papers listed since last one decade by Gallian [23] through his 'dynamic surveys'.

Graceful Labeling

In the year 1967, Rosa introduced a new type of graph labeling which he named as β-labeling or one given by Graham and Sloane [31] in 1980. Let G be any graph and q be the number of edges in G. Rosa introduced a function f from the set of vertices of G to the set of integers{ 0, 1, 2, . . . , q }, so that each edge uv is assigned the label $|f(u) - f(v)|$, with all labels distinct. It was believed that these labeling would help solve Ringel's conjecture [64], which involves decomposing a complete graph into isomorphic sub graphs. Golomb [30] independently studied the same type of labeling and named this labeling as graceful labeling [2, 33, 38, 39, 41, 43, 47, 51, 84].

Gnanajothi [29] defined and studied odd graceful graphs. The challenge in odd graceful labeling is to find out whether a given graph is odd graceful, and if it is then how to label the vertices. The common approach in proving the odd gracefulness of special classes of graphs is to either provide formulas for odd graceful labeling in the given graph, or construct desired labeling from combiningthe famous classes of odd graceful graphs.

Since then, different types of graceful labeling of graphs have been introduced and studied extensively by several graph theorists. Some of them are even graceful labeling [49, 60, 69], odd-even graceful labeling [57], edge graceful labeling, k -edge graceful labeling [24, 74], edge-odd graceful labeling, even-edge graceful labeling and k - even-edge graceful labeling [27 and 42].

Multiprotocol Label Switching

The Multiprotocol Label Switching (MPLS) working group was first formed in 1997. The First MPLS Layer 3 Virtual Private Networks (L3VPN) and Traffic Engineering (TE) deployment was done in 1999. Recently, MPLS transport profile was introduced in 2011. Before MPLS, a number of different technologies were deployed such as Frame Relay and Asynchronous Transfer Mode (ATM). Frame Relay and ATM uses frames or cells throughout a network.

MPLS technologies have evolved with the strengths and weaknesses of ATM in mind. Many network engineers have agreed that ATM and Frame Relay will be replaced with a protocol that requires less overhead, providing connection-oriented services for variable-length frames. MPLS has been replacing some of these technologies in the marketplace. It is highly possible that MPLS will completely replace these technologies in the future, thus aligning these technologies with current and future technology needs [55, 61 and 66].

MPLS uses labels in the packets to transport the data. It allows the core network devices to switch the data packets instead of looking at the destination IP address in the routing table. MPLS label path is pre-established from source to destination. MPLS data packets can be run on other layer 2 technologies such as Frame relay, ATM, Ethernet.

MPLS improves the performance of data packet forwarding in the network. Itprovides highly scalable network which is topology driven instead of flow driven. It supports Quality of service (QOS) by prioritizing the critical applications and processing them faster. MPLS network has the capability to restore the failed connections at very high speed than the traditional network.

Objectives and Scope of the Study

The objectives of this research work are the following:

i) To study the different types of graceful labeling and find new classes ofgraceful graphs.
ii) To investigate the possibilities of introducing new types of graceful labeling and find the corresponding classes of graceful graphs.
iii) To investigate the relations among different types of graceful labeling.
iv) To investigate the relation between different types of graceful labeling andother labeling.
v) To study the applications of the different types of graceful labeling.

This research work will introduce definitions for new types of graceful graphs in graph theory and also new classes of graceful graphs will be established. Further, the study of relation between different types of graphs, graceful or otherwise will open a new branch of study in graph theory. This research has enormous scope for further studies.

Organization of the Thesis

This thesis contains six chapters. The organization of the contents in each of these chapters is given below.

The **first chapter** of the thesis contains a brief introduction to graph labeling, graceful labeling and their applications. Literature review is carried out briefly. This chapter also contains the objectives and the scope of the thesis.

The **second chapter** provides the basic definitions and theorems on graphs, graph labeling, graceful labeling and Multiprotocol Label Switching communication networks, which are needed for discussion in the subsequent chapters.

The **third chapter** is divided into three sections as follows. The first section illustrates some of the results on odd graceful labeling. Odd gracefulness of the graph $\langle K_{1,n}, K_{1,m}: w \rangle \forall n, m,$ the corona graph $C_4(K_{1,n})$, and $k1$, the Dutch windmill graph m,D4 when m 2 are discussed.

The second section investigates the even gracefulness of the spider graph , ,2 S n n " the coconut tree CT(n,m)"n,m, the graph , : , , 1, 1, K K w n m n m " all caterpillar graph, the corona graph (), 4 1,n C K where n ³ 1, the shadow graph ( ), 2 n D P where n ³ 2, the shadow graph D K n n ( )" 2 1, and the graph , * n+k P where n ³ 2 and k ³ 1.

The inferences from the above studies on graceful labeling motivated the development of a new labeling technique termed as **even-even graceful labeling.** This study is carried out in the third section.

Definition A (p,q)-graph G is said to be even-even graceful if there is a bijection f from the edge set E(G) to the set { 2, 4,. . . , 2q } such that the induced mapping * f from the vertex set V(G) to the set { 0, 1, 2, . . . , (2k −1)} defined by f (v)= (Σ f (uv))(mod 2k) *, sum taken over all edges incident to v , where k = max{p, q} makes all edge labels distinct. It has been proved that the following well known families of graphs namely

(i) path graph P n n (ii) star graph n K 1, when n is even (iii) the graph P n n (4) (iv) the graph K K w n n , : 1, 1, n (v) perfect m -ary tree when m is even (vi) cycle graph n C when n is odd (vii) wheel graph n W when n 0(mod 4) (viii) dumbbell graph $D(n, n)$ (ix) the graph, n n C where n 3 and n is odd integer (x) Cartesian product graph P C n n are all even-even graceful.

The **fourth chapter** begins with a detailed study on the relation between different types of graceful labeling. Further, the relation between even-even graceful labeling and other labeling such as E -cordial labeling, totally magic cordial labeling, multiplicative labeling, strongly multiplicative labeling and multiplicative divisor labeling are investigated. The concept of complementary edge-odd graceful labeling has been introduced with an example.

The **fifth chapter** introduces a new concept of *k* **- even-even edge graceful labeling** of a graph and derives some family of graphs that are *k* -even-even edge graceful. Many variations and generalizations of labeling of graphs have been studied by many authors in many ways. Graceful labeling of graph has a lot of applications. In this chapter an application of *k* -even-even edge graceful labeling is also established.

Definition A (*p*,*q*)-graph *G* is said to be *k* **-even-even graceful** (*k* >0) if there is a bijection *f* from the edge set *E*(*G*) to the set (2*k*, 2*k* 2, . . . , 2*k*, 2*q*, 2) such that the induced mapping **f* from the vertex set *V*(*G*) to the set (0, 2, . . . , 2*z* -2) defined by *f* (*v*) = (c*f* (*uv*) (mod 2*z*) *, sum taken over all edges incident to *v* , where *z* = max{*p*,*q*}makes all edge labels distinct.

It shows that, the well known graphs namely the star graph *n K* 1, when *n* is even and *k* = 1(mod *n* = 1), the friendship graph *n F* when *n* is odd, the prism graph , *n Y n* > -3 and the Cartesian product graph , *m n C .C* where *n*=3 and *m* ≡1(mod 4) and the corona graph ( ), *m* 1,*n C K* when *m* is odd, *n* is even and *m* divides *n* are *k* **-** even-even edge graceful. Further, an application of *k* -even-even edge graceful labeling in MPLS (Multiprotocol Label Switching) communication networks is carried out. A graph*G* = (*V* , *E*) represents a MPLS communication network. The vertices are routers and the edges are typically the links between routers.

In a MPLS communication network, it is useful to assign an even label toeach destination IP networks. Labels are assigned and distributed before arrival of data traffic. This means that if a route exists in the IP forwarding table, a label has already been allocated for the route and so traffic arriving at a router can be label swapped immediately.

The **sixth chapter** contains the summary of this research work. The promising areas of future research are also discussed in this chapter.

Conclusion

This chapter comprises a general introduction to graph theory, graph labeling, graceful labeling and multiprotocol label switching. The objectives of this research work and the scope of the study are given explicitly. Further, a brief discussion on the organization of the thesis is carried out chapter wise.

The basic concepts in graph theory frequently used in this thesis and also a brief review of literature are given in the next chapter.

Summary

The comprehensive goal of this study is to scrutinize the amplification of graph labeling, graceful labeling and various types of graceful labeling. This thesis employs a mathematical approach comprising aggregated and solid level of analysis using recognized data to explore the relationship between various graph labeling of graphs. Further it throws a special focus on how *k* -even-even edge graceful labeling is applied to Multiprotocol Label Switching communication network.

One of the objectives of this research work is to study the different types of graceful labeling and find new classes of graceful graphs. Keeping the basic concepts and ideas of graph theory, constructions are given and the following theorems are established. The odd gracefulness of the graph ,,:,1,1,mnwKK mn ∀ the corona graph), (1,4 n KC where 1 ≥n, the coconut tree , ,)(, m nmnCT ∀ the graph , ∗ + n k P where 2 ≥n and 1 ≥k , the Dutch windmill graph ( ) mD 4 when 2 ≥m is successfully derived. Even gracefulness of the spider graph , ,2 nS n ∀ the coconut tree ,,) (, mnmnCT ∀ the graph , , :, 1,1, mnwKK mn ∀ all caterpillar graph, the corona graph), (1,4 n KC where 1, ≥n the shadow graph), (2 n PD where 2 ≥n , the shadow graph n KD n ∀) (1,2 and the graph , ∗ + n k P where 2 ≥n and 1 ≥k has been established. Also, a new type of graceful labeling called even-even graceful labeling is introduced and the corresponding classes of graceful graphs are given. The path graph 1),(2 1 ≥+ nP n the star graph n K 1, when n is even, the graph (4) n P for every n, the graph wKK nn : , 1,1, for every n, perfect m-ary tree when m is even, the cycle graph n C when n is odd, the wheel graph n W when 4) (mod0≡n , dumbbell graph (,) nnD for every n , the graph ( ) n n C where 3 ≥n and n is odd and the Cartesian product graph nCP n∀×2 are even-even graceful. There may be many more classes of graphs that are even-even graceful.

Further the relations among different types of graceful labeling and also the relation between different types of graceful labeling and other labeling are investigated. A graph has an odd-even graceful labeling if and only if it has a graceful labeling. In addition, the study established that graceful graphs are even graceful. On the other hand, even graceful graphs are graceful if they have an even graceful labeling whose vertex labels are all even.

The relation between different types of graceful labeling and other well known types of labeling namely graceful labeling and signed product cordial labeling of a tree, even graceful labeling and) (k n + -equitable labeling of the graph , ∗ + n k P even-even graceful labeling and E -cordial labeling of a perfect m-ary tree, even even graceful labeling and totally magic cordial labeling of a tree, even-even graceful and multiplicative labeling of an m-ary tree, even-even graceful and strongly multiplicative labeling of the m-ary tree and even-even graceful and modular multiplicative divisor labeling of the m-ary tree are established. The concept of complementary edge-odd graceful labeling is introduced.

This thesis further defines a new labeling technique namely k -even-even edge graceful labeling. The star graph n K 1, when n is even and 1), (mod1 +≡ nk the friendship graph n F when 0 >k and n is odd, the prism graph , n Y where 0,>k 3 ≥n and the Cartesian product graph , nm CC × where 0, >k 3 =n and 4)(mod1≡m and the corona graph), (1, nm KC when 0, >k m is odd, n is even and m divides n are k -even even edge graceful.

There are several papers on graph labeling that observed and identified its usage towards communication network. MPLS is a technology in high performance telecommunication network that sends data packet from one network node to the next network node, depending on the short path labels instead of looking at the long network IP addresses, thus avoiding complex lookups in an IP routing table. This thesis demonstrates how k -even-even edge graceful labeling is applied to Multiprotocol Label Switching communication networking. This concept is used to create a unique label to each destination network, which enables one to do the communications at very high speed in the modern technology.

Analogous study of other graph families, their different labeling techniques and its purpose is an open area of research.

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