Studies on Hamiltonian Colorings of Graphs

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Objectives and Planes

This dissertation under the title "On the Recent Colorings of Graphs" is in the field of graph theory. The contents of this thesis may be conveniently divided into five chapters, in which the first is the introductory chapter, the second chapter presents many results on T - colorings and T -graphs. The third one discusses and presents new results on L(2, 1) - colorings and Radio colorings of graphs. Fourth deals with the study of Hamiltonian colorings of graphs. The lastchapter contains many further results on Hamiltonian colorings of graphs.

Chapter-1 begins with the objectives and planes, followed by basic definitions and notations, needed for the subsequent chapters.

Chapter-2, in Section 2.1, we discuss and present new results and bounds on the T - colorings of graphs and certain related concepts like: complementary coloring of T - colorings, T -chromatic numbers of graphs, c -spans, andT -spans. In Section 2.2, we study and present results on T -graphs and graph homomorphisms, properties of graph homomorphisms, weakly perfect graphs and it related conjecture. Further, we develop the relationship between span of a complete graph and the clique size of the T -graphs. In Section 2.3 we present theorems and examples on weakly perfect graphs with fixed chromatic numbers. Chapter-3, Section 3.1, we study and present results on L(2, 1) -colorings of graphs and certain related concepts like: c -span and L -span of L(2, 1) -colorings of graphs. Also, we present some upper bounds for L -span of graphs and their related conjecture. In Section 3.2, Radio colorings of graphs, the complementary colorings of radio coloring of graphs, and k -radio chromatic numbers are determined for connected graphs having fixed diameter. It is shown that certain properties with simple upper bounds exist for rcl(G). In Section 3.3, we discuss and present results on Radio antipodal colorings of graphs, antipodal chromatic numbers of paths.

Also, we present a sufficient condition for the antipodal chromatic number of a connected subgraph of a connected graph G of diameter d to be bounded above by ac(G). Finally, an upper bound for the antipodal chromatic number of paths determined. In Section 3.4, various bounds for antipodal chromatic numbers of graphs are presented.

In Chapter-4, we study and present new results on hamiltonian colorings of graphs, hamiltonian chromatic number of graphs. Further the minimum hamiltonian coloring of graphs, graphs with equal hamiltonian chromatic number and antipodal chromatic number of graphs studied in detail. Some bounds for the hamiltonian chromatic numbers of graphs are presented.

Basic Definitions and Notations

A graph G is a finite nonempty set V of objects called vertices, together with a set E of 2 -element subsets of V called edges. Each edge {u, v} of V is generally denoted by uv (or vu.) If e = uv, then the edge e is said to join u and v . The number of vertices in a graph G is the order of G and the number of edges is the size of G. To indicate that a graph G has vertex set V and edge set E, we sometimes write G = (V, E). To emphasize that V is the vertex set of a graph G, we often V as V(G). For the same reason, we also write E as E(G). A graph of order 1 is called a trivial graph and a nontrivial graph has two or more vertices.

If uv is an edge of a graph G, then u and v are adjacent vertices in G . Two adjacent vertices are referred as neighbors of each other. If uv and vw are distinct edges in a graph G, then uv and vw are adjacent edges in G . The vertex u and the edge uv are said to be incident with each other. Similarly v and uv are incident.

The degree of a vertex v in a graph G is the number of vertices in G that are adjacent to v, and is denoted by degG(v) (or deg(v).) A vertex of degree 0 is referred as an isolated vertex and a vertex degree 1 is an end-vertex. An edge incident with an end-vertex is called a pendent edge. The largest degree among the vertices is called the maximum degree of G is denoted by (G). The minimum degree of G is denoted by δ(G). Thus if v is a vertex of a graph G order n, then $0 \leq \delta(G) \leq \deg(v) \leq (G) \leq n - 1$.

A graph G is finite if both its vertex set and edge set are finite. A graph G is simple if it has no loops(i.e., edges having identical ends) and no two of its edges join the same pair of vertices. A graph H is said to be a subgraph of a graph G if V(H) ⊆ V(G) andE(H) ⊆ E(G). If V(H) = V(G), then H is called a spanning subgraph of G. If H is asubgraph of G and either V(H) is a proper subset of V(G) or E(H) is a proper subset of E(G), then H is said to be a proper subgraph of G. For a nonempty subset S of V(G), the subgraph G[S] of G induced by S has S as its vertex set and two vertices u and v of S are adjacent in G[S] if and only if u and v are adjacent in G. A subgraph H of a graph G is called induced subgraph of G if there a nonempty subset S of V(G) such that H = G[S]. Thus G[V(G)] = G. For a nonempty set X of edges of a graph G, the subgraph G[X] induced by X has X as its edges set and its vertex set belongs to the vertices of X. A nontrivial graph G is called a bipartite graph if it is possible to partition V(G) into two non-empty subsets U and W (called partite sets) such that every edge of G joins a vertex of U to a vertex of W. A bipartite graph having partite sets U and W is called a complete bipartite graph if every vertex of U is adjacent to every vertex of W, then this complete bipartite graph is denoted by Ks,t (or Kt,s). The graph K1,t is called a star.

A walk in a graph G is a finite non-empty sequence W = v0, e1, v1, e2, · · · ,ek, vk whose terms are alternately vertices and edges, such that $1 \leq i \leq k$, the ends of ei are vi−1 and vi . A walk whose initial and terminal verticesare distinct is an open walk; otherwise, it is a closed walk. A walk of a graph G in which no edge is repeated is called a trail in G. A walk of a graph G in which no vertex and no edges repeated is called a path. Any closed path is called a cycle. A nontrivial closed walk of a graph G in which no edge is repeated is called a circuit in G. Two vertices u and v in a graph G are said to be connected if G contains a path connecting u and v i.e., (u, v) - path. A graph G is said to be connected if every two vertices of G are connected. A graph G is not connected is called a disconnected graph. A connected graph without cycles is called a tree.

Let G be a nontrivial connected graph. A circuit of G that contains every edge of G is called an Eulerian Circuit, while an open trail that contains every edge of G is an Eulerian trail. A connected graph G is called Eulierian if G contains anEulireian circuit. Let G be a graph, a path in G that contains every vertex of G is called a Hamiltonian path of G, while a cycle in G that contains every vertex of G is called a Hamiltonian cycle ofG. A graph that contains a Hamiltonian cycle is called Hamiltonian graph.

The distance d(u, v) from a vertex u to vertex v in a connected graph G is the minimum length of the (u, v) -path in G. A (u, v) -path of length d(u, v) is called a (u, v) - geodesic. The distance function d defined above satisfies the following properties in a connected graph G :

(1) d(u, v) ≥ 0 for every two vertices u and v of G
(2) d(u, v) = 0 if and only if u = v for all u, v ∈ V(G)
(3) d(u, v) = d(v, u) foe all u, v ∈ V(G) (the symmetric property)
(4) d(u, w) ≤ d(u, v) + d(v, w) for all u, v, w ∈ V(G) (the triangle inequality).

Since dsatisfies the above (4) properties dis a metric on V(G). (V(G), d) forms a metric space.Since dis symmetric, we can speak of the ”distancebetween two vertices u and v ” rather than the ” distance from v to u ”

The eccentricity e(v) of a vertex v in a connected graph G is the distance between v and a vertex farthest from v in G. The diameter of a connected graph G denoted by diam (G) of G is the greatest eccentricity among the vertices of G, while the radiusof G denoted by rad (G) is the smallest eccentricity among the vertices of G. The diameter of G is also the greatest distance between any two vertices of G. A vertex v with e(v) = rad(G) is called a central vertex of G and a vertex v with e(v) = diam(G) is called a peripheral vertex of G. Two vertices u and v of a graph G with d(u, v) = diam(G) are called the antipodal vertices of G. If u and v are antipodal vertices in G, then each of u and v is called a peripheral vertex of G .

A proper vertex coloring of a graph G is an assignment of colors to the vertices ofG, such that adjacent vertices of G are colored differently. A graph G is a k -colorable,if there exists a proper coloring of G from the set of k colors. In other words, G isk -colorable if there exists a k -coloring of G. The minimum positive integer k for whichG is k -colorable is called the chromatic number of G and is denoted by χ(G).Additional definitions, results or notations will be introduced as the need arises.

On Hamiltonian Colorings of Graphs

Introduction

The concepts Radio k -colorings and radio k -chromatic number of graphs were inspired by the so-called channel assignment problem, where channels are assigned to FM radio stations according to the distances between the stations (and some other factors as well). Since Radio 1-chromatic number is the chromatic number χ(G), radio k -colorings provide a generalization of ordinary colorings of graphs. The radio d -chromatic number was studied in the previous chapter and was also called the radio-number. Radio d -colorings are also referred to as radio labelings since no two vertices can be colored the same in a radio d -coloring. Thus, in a radio-labeling of a connected graph of diameter d, thelabels (colors) assigned to adjacent vertices must differ by at least d, the labels assigned to two vertices whose distance is 2 must differ by at least d − 1, and so on, up to the vertices whose distance is d, that is, antipodal vertices, whose labels are only required to be different. A radio(d − 1) -coloring is less restrictive in that colors assigned to two vertices whose distance is i, where1 ≤ i ≤ d, are only required to differ by at least d − i. In particular , antipodal vertices can be colored the same. For this reason, radio (d − 1) - colorings are also called radio antipodal colorings or, more simply, antipodal colorings. Antipodal colorings of graphs were studied in the previous chapter, where rcd−1(G) was written as ac(G). Radio k -coloring of paths were studied in [6] for all possible values of k. In the case of an antipodal coloring of the path Pn of order n ≥ 3 (and diameter n − 1), only the end-vertices of Pn are permitted to be colored the same since the only pair of antipodal vertices in Pn are its twoend-vertices. Of course, the two end-vertices of Pn are connected by a hamiltonian path. As mentioned earlier, if u and v are any two distinct vertices of Pn and d(u, v) = i, then $|c(u) - c(v)| \geq n - 1 - i$. Since Pn is a tree, not only is i the length of a shortest u − v path in Pn, it is the length of any u − v path in Pn since every two vertices are connected by a unique path. In particular, the length of a longest u − v path in Pn is i as well. For vertices u and v in a connected graph G, let D(u, v) denote the length of a longest u − v path in G. Thus for every connected graph G of order n and diameter d, both d(u, v) and D(u, v) are metrics on V(G). Radio k -colorings ofG are inspired by radio antipodal colorings c which are defined by the inequality$d(u, v) + |c(u) - c(v)| \geq d$. If G is a path, then $d(u, v) + |c(u) - c(v)| \geq d$. is equivalent to$D(u, v)+|c(u)-c(v)| \geq n- 1$, which suggests an extension of the coloring c that satisfies$D(u, v)+ |c(u)-c(v)| \geq n - 1$, for an arbitrary connected graph G.

Definition Let G be a Connected Graph Of order n.

1. A hamiltonian coloring c of G is an assignment of colors (positive integers) to the vertices of G such that $D(u, v) + |c(u) - c(v)| \geq n - 1$, for every two distinct vertices u and v of G. In hamiltonian coloring of G, two vertices u and v can be assigned the same color only if G contains a hamiltonian u − v path.
2. Thevalue hc(c) of a Hamiltonian coloring c of Gisthe maximum color assigned to a vertex of G.
3. The hamiltonian chromatic number hc(G) of G is min {hc(c)} over all hamiltonian colorings c of G.
4. A hamiltonian coloring c of G is a minimum hamiltonian coloringif hc(c) = hc(G).

Definition A graph G is hamiltonian-connected if for every pair u, v of distinct vertices of G, there is a hamiltonian u − v path.

Consequently, we have the following fact:

Proposition Let G be a connected graph. Then hc(G) = 1 if and only if G is hamiltonian-connected.

In a certain sense, the hamiltonian chromatic number of a connected graph G measures how close G is to being hamiltonian-connected, the nearer the hamiltonian chromatic number of a connected graph G is to 1, the closer G is to being hamiltonian-connected.

Graphs with Equal Hamiltonian Chromatic Numbers and Antipodal Chromatic Numbers

Since the path Pn is the only graph G of order n for which diam G = n − 1, we have the following fact:

Proposition . If G is a path, then hc(G) = ac(G).

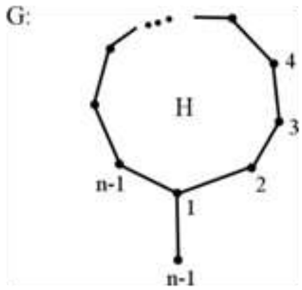
Earlier it was shown that $ac(Pn) \leq n-21 + 1$ for every positive integer n . Moreover, it was shown in [6] that $ac(Pn) \leq n-21 - (n - 1)/2 + 4$ for odd integers $n \geq 7$. Therefore, we have the following corollary:

Corollary . For every positive integer n, $hc(Pn) \leq n-21 + 1$. Furthermore, for all odd integers $n \geq 7$, $hc(Pn) \leq n-21 - n-21 + 4$.

In order to see that the converse of Observation 4.2.1 is false, we first consider the following lemmas.

Lemma. Let H be a hamiltonian graph of order $n - 1 \geq 3$. If G is a graph obtained from H by adding a pendant edge, then hc(G) = n − 1.

Proof . Let C : v1, v2, · · · , vn−1, v1 be a hamiltonian cycle of H and let v1vn be the pendant edge of G. Let c be a hamiltonian coloring of G. Since $D(u, v) \leq n - 2$ for allu, v ∈ V(C), there is no pair of vertices in C that are colored the same by c. This implies that $hc(c) \geq n - 1$ an d so $hc(G) \geq n - 1$. Define a coloring c0 of G by c0(vi) = i for $1 \leq i \leq n - 1$ and c0(vn) = n − 1 (see Figure 4.1). We show that c0 is a hamiltonian coloring of G.

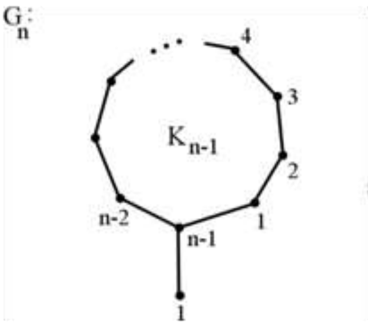


(A hamiltonian coloring c0 of G)

First consider two vertices vi and v j, where $1 \leq i < j \leq n - 1$. Then$|c0(vi)-c0(v j)|= j - i$, while D(vi, v

$j) \geq n-1+i-j$. Thus$|c_0(v_i) - c_0(v_j)| + D(v_i, v_j) \geq n - 1$. Now consider the two vertices v_i and v_n, where $1 \leq i \leq n-1$. Then $|c_0(v_i) - c_0(v_n)| = n - 1 - i$, while$D(v_i, v_n) \geq i$. Hence $|c_0(v_i) - c_0(v_n)| + D(v_i, v_n) \geq n - 1$. Therefore, c_0 is a hamiltonian coloring of G and so $hc(G) \leq hc(c_0) = n - 1$.

For $n \geq 4$, let G_n be the graph obtained from the complete graph K_{n-1} by adding apendant edge. Then G_n has order n and diameter 2. Let $V(G_n) = \{v_1, v_2, ..., v_n\}$, where $\deg v_n = 1$ and $v_{n-1}v_n \in E(G)$. By Lemma 4.2.1, $hc(G_n) = n - 1$. We now show that $ac(G_n) = hc(G_n) = n - 1$. Let c be an antipodal coloring of G_n. Since $diam G_n = 2$, it follows that the colors $c(v_1), c(v_2), ..., c(v_{n-1})$ are distinct and so $ac(G_n) \geq n - 1$. Moreover, the coloring c of G_n defined by $c(v_i) = i$ for $1 \leq i \leq n - 1$, $c(v_n) = 1$ is an antipodal coloring of G_n (see Figure 4.2) and so $ac(G_n) = n - 1$.

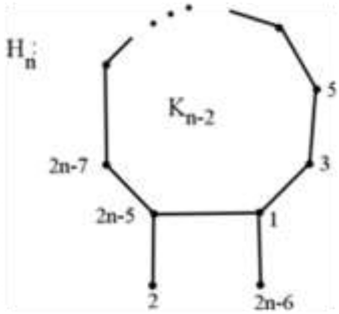


(An antipodal coloring cj of Gn)

Hence there is an infinite class of graphs G of diameter 2 such that$hc(G) = ac(G)$. We now show that there exists an infinite class of graphs G of diameter 3 such that $hc(G) = ac(G)$.

Lemma 4.2.2. For $n \geq 5$, let H_n be the graph obtained from the complete graph K_{n-2}, where $V(K_{n-2}) = \{v_1, v_2, \cdots, v_{n-2}\}$, by adding the two pendant edges v_1v_{n-1} and $v_{n-2}v_n$. Then H_n is a graph of order n and diameter 3 such that $hc(H_n) = ac(H_n) = 2n - 5$.

Proof 4.2.2. Let c be a hamiltonian coloring of H_n. Since $D(u, v) = n - 3$ for all $u, v \in V(K_{n-2})$, the colors of every two vertices of K_{n-2} must differ by at least 2, implying that $hc(c) \geq 2n - 5$ and so $hc(H_n) \geq 2n - 5$.



(A hamiltonian coloring c1 of Hn)

Define a coloring c_1 of H_n by $c_1(v_i) = 2i - 1$ for $1 \leq i \leq n - 2$, $c_1(v_{n-1}) = 2n - 6$, and $c_1(v_n) = 2$. (see Figure 4.3) We show that c_1 is a hamiltonian coloring of H_n. For vertices v_i and v_j, where $1 \leq i < j \leq n - 2$, it follows that $|c_1(v_i) - c_1(v_j)| = (2j - 1) - (2i - 1) = 2j - 2i$. Furthermore, $D(v_i, v_j) = n - 3$ and so $|c_1(v_i) - c_1(v_j)|+ D(v_i, v_j) = 2(j - i) + n - 3 \geq 2 + n - 3 = n - 1$. Next, we consider two vertices v_i and v_{n-1}, where $1 \leq i \leq n - 2$. In this case, $|c_1(v_i) - c_1(v_{n-1})| = (2n - 6) - (2i - 1) = 2n - 2i - 5$ if $1 \leq i \leq n - 3$, while $|c_1(v_{n-2}) - c_1(v_{n-1})| = 1$. Moreover, $D(v_1,v_{n-1}) = 1$ and $D(v_i,v_{n-1})= n - 2$ for$2 \leq i \leq n - 2$. Thus, for $1 \leq i \leq n - 3$,$|c_1(v_i) - c_1(v_{n-1})| + D(v_i, v_{n-1}) \geq (2n - 2i - 5) + (n - 2)= 3n - 2i - 7 \geq n - 1$;while $|c_1(v_{n-2}) - c_1(v_{n-1})| + D(v_{n-2}, v_{n-1}) = 1 + (n - 2) = n - 1$. Similarly, $c_1(v_i) - c_1(v_n)| + D(v_i, v_n) \geq n - 1$ for $1 \leq i \leq n - 1$. Hence c_1 is a hamiltonian coloring of H_n and so $hc(H_n) \leq hc(c_1) = 2n - 5$. Therefore, $hc(H_n) = 2n - 5$. We now show that $ac(H_n) = 2n - 5$ as well. Let c be an antipodal coloring of H_n. Since $diam H_n = 3$, it follows that the colors $c(v_1), c(v_2), \cdots, c(v_{n-2})$ differ by at least 2 and so $ac(H_n) \geq 2n - 5$. Since the coloring c_1 of H_n shown in Figure 4.3 is also an antipodal coloring of H_n,$ac(H_n) \leq 2n - 5$ and so $ac(H_n) = 2n - 5$.

Whether there exists an infinite class of graphs G that are not paths, whose diameter exceeds 3 and for which $hc(G) = ac(G)$, is not known. Indeed, it is not known if there is even one such graph that is not a path.

Hamiltonian Chromatic Numbers of Some Special Class of Graphs

Since the complete graph K_n is hamiltonian-connected, $hc(K_n) = 1$. We state this below for later reference:

Proposition . For $n \geq 1$, hc(Kn) = 1.

We now consider the complete bipartite graphs Kr,s, beginning withKr,r. The graph Kr,r has order n = 2r and is hamiltonian but is not hamiltonian-connected. For distinct vertices u and v of Kr,r,D(u, v) = □ □ n − 1 if uv∈ E(Kr,r)

Therefore, for a hamiltoniancoloring□ ofKr,r, every two nonadjacent vertices must be colored differently (while adjacent vertices can be colored the same). This implies thathc(Kr,r) = χ(Kr,r) = r. We now determine hc(Kr,s) with r < s, beginning withr = 1.

Theorem 4.3.1. For $n \geq 3$, hc(K1,n−1) = (n − 2)2 + 1.

Proof 4.3.1. Since hc(K1,2) = 2, the result holds for n = 3. So we may assume that $n \geq 4$. Let G = K1,n−1 with vertex set {v1, v2, · · · , vn}, where vn is the central vertex of G. Define a coloring c of G by c(vn) = 1 and c(vi) = (n − 1) + (i − 1)(n − 3) for $1 \leq i \leq n - 1$. Since c is a hamiltonian coloring, hc(G) ≤ hc(c) = c(vn−1) = (n − 1) + (n − 2)(n − 3) = (n − 2)2 + 1.

Next we show that hc(G) ≥ (n − 2)2 + 1. Let c be a minimum hamiltonian coloring of G. Since G contains no hamiltonian path, no two vertices can be colored the same.We may assume that c(v1) < c(v2) < · · · < c(vn−1). We consider three cases.

Case 1.c(vn) = 1. Since D(v1, vn) = 1 and D(vi, vi+1) = 2 for i with$1 \leq i \leq n - 2$, it follows that c(v1) ≥ n − 1 and c(vi+1) ≥ c(vi) + (n − 3) for all $1 \leq i \leq n - 2$. This implies that c(vn−1) ≥ (n − 1) +(n − 2)(n − 3) = (n − 2)2 + 1. Therefore, hc(c) = hc(G) ≥ (n − 2)2 + 1.

Case 2.c(vn) = hc(c). Then 1 = c(v1) < c(v2) < · · · < c(vn−1) < c(vn). For each i with $2 \leq i \leq n - 1$, it follows that c(vi) ≥ (n − 2) + (i − 2)(n − 3). In particular, c(vn−1) ≥ (n − 2) + (n − 3)(n − 3) = n2 − 5n + 7. Thusc(vn) ≥ c(vn−1) + (n − 2) ≥(n2 − 5n + 7) + (n − 2) = (n − 2)2 + 1. Therefore, hc(c) = hc(G) ≥ (n − 2)2 + 1.

Case 3.c(vj) < c(vn) < c(vj+1) for some j with $1 \leq j \leq n - 2$. Thus

c(vj) ≥ (n − 2) + (j − 2)(n − 3),

c(vn) ≥ c(vj) + (n − 2) = 2(n − 2) + (j − 2)(n − 3),

c(vj+1) ≥ c(vn) + (n − 2) ≥ 3(n − 2) + (j − 2)(n − 3).

This implies that c(vn−1) ≥ 3(n − 2) + (n − 4)(n − 3) = n2 − 4n + 6 > (n − 2)2 + 1. Hence, hc(c) = hc(G) ≥ (n − 2)2 + 1.We now consider Kr,s, where $2 \leq r < s$, with partite sets V1 and V2 such that |V1| = r and |V2| = s. Then□ 2r − 2 = n − s + r– 2if u, v ∈ V1, D(u, v) = □ □ □ 2r − 1 = n − s + r − 1 if uv ∈ E(K consequently, if c is a hamiltonian coloring of Kr,s (r < s),

thenr,s) $2r=n-s+r$ if$u,v\in V_2$

$s-r+1$ if$u,v\in V_1$,

|c(u) − c(v)| ≥ □ □ s − r if uv∈ E(K),r,ss − r − 1 if u, v ∈ V2.

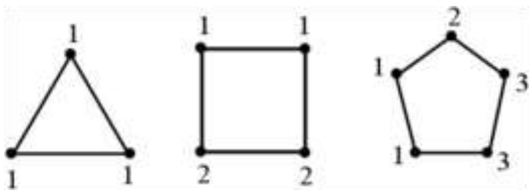
Theorem 4.3.2. For integers r and s with $2 \leq r < s$, hc(Kr,s) = (s − 1)2 − (r − 1)2.

Proof 4.3.2. Let V1 = {u1, u2, · · · , ur} and V2 = {v1, v2, · · · , vs} be the partite sets of Kr,s. Define a coloring c of Kr,s by c(ui) = 1 + (i − 1)(s − r + 1) for $1 \leq i \leq r - 1$, c(vj) = c(ur−1) + (s − r) + (j − 1)(s − r − 1) = (r − 1)(s − r + 1) + (j − 1)(s − r − 1) for $1 \leq j \leq s$, and c(ur) = c(vs) + (s − r) = (s − 1)2 − (r − 1)2. Since c is a hamiltonian coloring of Kr,s, it follows that hc(Kr,s) ≤ hc(c) ≤ (s − 1)2 − (r − 1)2.

A V2 -block of Kr,s is defined similarly. Let A1, A2, · · · , Ap ($p \geq 1$) be the distinct V1 -blocks of Kr,s such that if wj ∈ Ai and wjj ∈ Aj, where $1 \leq i < j \leq p$, then c(wj) < c(wjj). If $p \geq 2$, then Kr,s contains V2 -blocks B1, B2, · · · , Bp−1 such that for each integer i ($1 \leq i \leq p - 1$) and for wj ∈ Ai, w ∈ Bi, wjj ∈ Ai+1, it follows that c(wj) < c(w) < c(wjj).

The graph Kr,s may contain up to two additional V2 -blocks, namely, B0 and Bp such thatif y ∈ B0 and yj ∈ A1, then c(y) < c(yj); while if z ∈ Ap and zj ∈ Bp, then c(z) < c(zj). If p = 1, then at least one of B0 and B1 must exist. Hence Kr,s contains p V1 -blocks and p − 1 + tV2 -blocks, where t ∈ {0, 1, 2}. Consequently, there are exactly (1) r − p distinct pairs wi, wi+1 of vertices, both of which belong to V1, (2) 2p − 2 + t distinct pairs {wi, wi+1} of vertices, exactly one of which belongs to V1, and (3) s − (p − 1 + t) distinct pairs {wi, wi+1} of vertices, both of which belong to V2.

Since (1) the colors of every two vertices wi and wi+1, both of which belong to V1, mustdiffer by at least s − r + 1, (2) the colors of every two vertices wi and wi+1, exactly one of which belongs to V1, must differ by at least s − r, and (3) the colors of every two vertices wi and wi+1, both of which belong to V2, must differ by at least s − r − 1, it follows that c(wr+s) ≤ 1+(r−p)(s−r+1)+(2p−2+t)(s−r)+(s−(p−1+t))(s−r−1) = (s−1)2−(r−1)2+t.Since hc(Kr,s) ≤ (s − 1)2 − (r − 1)2 and t ≥ 0, it follows that t = 0 and that hc(Kr,s)=(s − 1)2−(r–1)



(Hamiltonian colorings of C3, C4 and C5)

We now determine the hamiltonianchromatic number of each cycle Minimum hamiltonian colorings of the cycles Cn for n = 3, 4, 5 are shown in Figure 4.4.

Definition 4.3.1. For a hamiltonian coloring c of a connected graph G, a set S = {u, v} of distinct vertices of G is called a c -pair if c(u) = c(v). We also write c(S) = c(u) = c(v).

Lemma 4.3.1. Let c be a minimum hamiltonian coloring of Cn, where n ≥ 4.

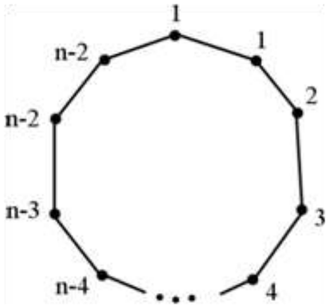
a) If u, v is a c -pair, then u and v are adjacent.
b) If S and S j are distinct c -pairs, then S ∩ S j = φ and c(S) ≠ c(S j).

Proof 4.3.3. If u and v are nonadjacent vertices of Cn, then D(u, v) < n − 1, implying that c(u) ≠ c(v) and so u and v are adjacent

To verify S ∩ S j = φ and c(S) ≠ c(S j), let S = {u, v} and S j = uj, vj be distinct c -pairs. Assume that S ∩ S j ≠ φ or c(S) = c(S j). If S ∩ S j ≠ φ, then we may assume that u ≠ uj and v = vj. This implies that c(u) = c(v) = c(vj) = c(uj) and therefore, {u, uj} is a c -pair as well. If c(S) = c(S j), then {u, uj} is also a c -pair. By (a), ({u, uj, v}) = C3, which is a contradiction.

Theorem 4.3.3. For n ≥ 3, hc(Cn) = n − 2.

Proof 4.3.4. Let Cn : v1, v2, · · · , vn, v1. Since hc(Cn) = n − 2 for n = 3, 4, 5, we may assume that n ≥ 6. Define a coloring c of Cn by c(v1) = n−2, c(v2) = 1, and c(vi) = i−2 for 3 ≤ i ≤ n (see Figure 4.5). Since c is a hamiltonian coloring, hc(Cn) ≤ n − 2.



(A hamiltonian coloring of Cn for n ≥ 6)

Next we show that hc(Cn) ≥ n−2. Let c be a minimum hamiltonian coloring of Cn and let q be the number of distinct c -pairs. Since hc(Cn) ≥ n−2, it follows that q ≥ 2. Denote these q c -pairs by S 1, S 2, · · · , S q. By Lemma 4.3.1(b), for all i, j with 1 ≤ i ≠ j ≤ q, we have S i T S j = φ. and c(S i) ≠ c(S j). If q = 2, then hc(c) ≥ n − 2; so we assume that q ≥ 3. Without loss of generality, we may assume that c(S 1) < c(S 2) < · · · < c(S q). For each i with 1 ≤ i ≤ q − 1, let Ai = {u ∈ V(G) : c(S i) < c(u) < c(S i+1)}. There exist nonnegative integers a1, a2, · · · , aq−1 such that |Ai| = c(S i+1) − c(S i) − 1 − ai for each integer i (1 ≤ i ≤ q − 1). Define a = a1 + a2 + · · · + aq−1 and I = {i :ai = 0, where 1 ≤ i ≤ q − 1}. At most c(S 1) − 1 vertices of Cn are assigned a color less than c(S 1) and at most hc(c) − c(S q) vertices of Cn are assigned a color exceeding c(S q). Since alln vertices of Cn are assigned a color by c, it follows that

n ≤ (c(S 1) − 1) + |S 1| + |A1| + |S 2| + |A2| + · · · + |Aq−1| + |S q| + (hc(c) − c(S q))qq−1

= (c(S 1) − 1) + (hc(c) − c(S q)) + X |si| + X |Ai|i=1i=1q−1

= (c(S 1) − 1) + (hc(c) − c(S q)) + 2q + X(c(si+1) $_{i=1}$c(si) − 1 − ai)

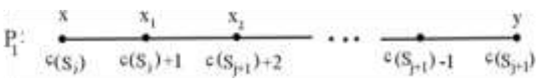
= hc(c) + q − a.

Since hc(c) ≤ n − 2, we get a ≤ q − 2. This implies that I ≠ φ. and so aj = 0 for some j with 1 ≤ j ≤ q − 1 and j ∈ I. Let S j = {x, xj} and S j+1 = {y, yj}. By Lemma 4.3.1(a), xxj, yyj ∈ E(Cn). Then Cn − xxj − yyj consists of two nontrivial paths P1 and

P2. Assume, without loss of generality, that x and y are the end-vertices of P1 and thus xj and yj are the end-vertices of P2. Since $j \in I$, there exists a vertex x1 of Cn such that $c(x1) = c(S\ j) + 1$. Since $|c(x1) - c(x)| = 1 = |c(x1) - c(xj)|$, we have $D(x1, x) \geq n - 2$ and $D(x1, xj) \leq n - 2$. It follows that x1 is adjacent to either x or xj , say x.

Now, let $n = 2k$ or $n = 2k + 1$ for some $k \geq 3$, according to whether n is even or n is odd. We claim that $c(S\ j+1) - c(S\ j) \geq k - 1$, for suppose that $c(S\ j+1) - c(S\ j) \leq k - 2$. If $c(S\ j) + 1 = c(S\ j+1)$, then $y = x1$. If $c(S\ j) + 1 < c(S\ j+1)$, then let x2 be a vertex of Cn such that $c(x2) = c(S\ j) + 2$. Then $D(x2, x1) \geq n - 2$; while $D(x2, x) \geq n - 3$, and $D(x2, xj) \geq n - 3$. This implies that x2 is adjacent to x1. Continuing in this manner, we see that P1 has length $c(S\ j+1)-c(S\ j)$ and its vertices are colored by c as shown in Figure 4.6. It is clear that P2 has length $n - 2 -(c(S\ j+1) - c(S\ j))$. Since $c(S\ j+1) - c(S\ j) \leq k - 2$, we get

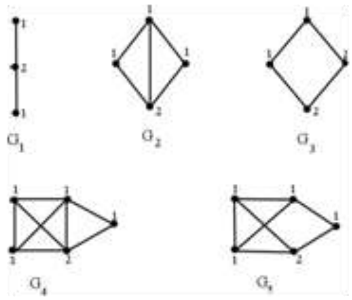
$D(xj, yj) = n-2-(c(S\ j+1)-c(S\ j))$. Thus $|c(xj)-c(yj)|+ D(xj, yj) = n-2$, which contradicts the fact that c is a hamiltonian coloring of Cn. Thus we have $c(S\ j+1) - c(S\ j) \geq k - 1$, as claimed.



(The coloring of P1)

Let y1 be a vertex such that $c(y1) = c(S\ j+1) - 1$. We see that y1 is adjacent to either yj or y. We claim that y1 is adjacent to y, for suppose that y1 is adjacent to yj. Then y1 belongs to P2. Recall that $aj = 0$. Since x1 belongs to P1 and the paths P1 and P2 have no common vertex, there exist vertices $x*$ and $y*$ of Cn such that $c(x1) \leq c(x*)$, $c(x*)+1 = c(y*)c(y1)$ and that $x* \in P1$ and $y* \in P2$. Hence $D(x*, y*) \leq n-3$, a contradiction. Thus y1 is adjacent to y.

We can therefore find vertices $x2, \cdots, xk-2$ of Cn such that $c(x2) = c(S\ j) + i$ for $i = 2, \cdots, k - 2$ and Px : $x, x1, x2, \cdots, xk-2$ is a sub path of P1. Similarly, we can find vertices $y2, \cdots, yk-2$ of Cn such that $c(y2) = c(S\ j+1) - i$ for $i = 2, \cdots, k - 2$ and that Py : $yk2, \cdots, y2, y1, y$ is a sub path of P1. We claim that Px and Py are not vertex-disjoint, for suppose that they are. Then since $q \geq 3$, it follows that $n \geq 2k + 2$, a contradiction. Thus Px and Py have a common vertex. This implies that the path P1 contains exactly one vertex colored i for each i with $c(S\ j) \leq i \leq c(S\ j+1)$ and has no vertices of any other color (see Figure.4.6 for the coloring of the vertices of P1). Therefore, the length of P1 is $c(S\ j+1) - c(S\ j)$ and the length of P2 is $n - 2 - (c(S\ j+1) - c(S\ j))$.



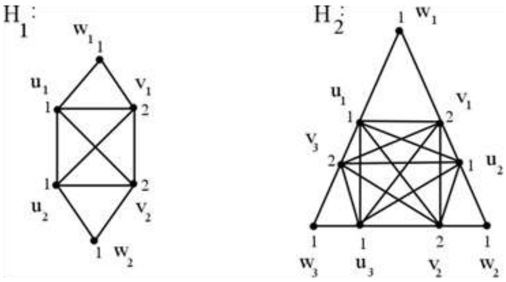
(Graphs of order n($3 \leq n \leq 5$) having hamiltonian chromatic number 2)

Recall that $c(S\ j+1) - c(S\ j) \geq k - 1$. If, in addition, $l \in I$, where $l \neq j$, then, as above, $c(S\ l+1) - c(S\ l) \geq k - 1$ and there is a path Q1 of length $c(S\ l+1) - c(S\ l)$ whose vertices are colored by $c(S\ l), c(S\ l) + 1, \cdots, c(S\ l+1)$ and where necessarily, Q1 is a proper sub path of P2. Assume, without loss of generality, that $l > j$. Let $S\ l = \{z, zj\}$ and $S\ l+1 = \{w, wj\}$. Then the sets S j, S j+1, S l, S l+1 are distinct except possibly S j+1 = S l. Since $Cn - xxj - yyj - zzj - wwj$ consists of at least three nontrivial paths, the lengths of at least two of which, namely P1 and Q1, are at least $k - 1$, it follows that Cn has at least $2(k - 1) + 4 = 2k + 2$ edges, which is impossible. Thus $I = \{ j\}$. Recall that $a \leq q - 2$. Since $|I| = 1$, it follows that $a \geq q - 2$. Hence $a = q - 2$. Since $n \leq hc(Cn) + q - a$, we have $n \leq hc(c) + 2$.So $hc(c) \geq n - 2$, as desired.

Hamiltonian Chromatic Numbers of Graphs Having given Orders

In this section we shall assume that we are considering connected graphs of order n for some fixed integer $n \geq 3$. We have already mentioned that a graph G has hamiltonian chromatic number 1 if and only if G is hamiltonian-connected. We now show that it is possible for a graph G to have hamiltonian chromatic number 2. All the graphs (of orders 3 to 5) shown in Figure 4.7 have hamiltonian chromatic number 2.

The graphs G2 and G4 (or G3 and G5) are actually special cases of a more generalclass of graphs. For $n \geq 4$, let G_{2n-6} be the graph of order n obtained by joiningtwo vertices u and v of K_{n-1} to a new vertex w and let $G_{2n-5} = G_{2n-6} - uv$. Then $hc(G_{2n-6}) = hc(G_{2n-5}) = 2$ for all $n \geq 4$. We also have other graphs of order n with hamiltonian chromatic number 2 if n is sufficiently large. The graphs H1 and H2 of have hamiltonian chromatic number 2.



(other graphs with hamiltonian chromatic number 2)

In general, for $n = 3k \geq 6$, let H_{k-1} be the graph obtained from K_{2k}, where $V(K_{2k}) = \{u_1, v_1, u_2, v_2, \cdots, u_k, v_k\}$, by adding the k new vertices $w_1, w_2, \cdots, w_k$ and joining w_i to u_i and v_i for $1 \leq i \leq k$. Then $hc(H_{k-1}) = 2$ for all $k \geq 2$. A hamiltonian coloring of H_{k-1} assigns 1 to u_i and w_i and 2 to v_i for all $i(1 \leq i \leq k)$. This class of examples shows that there exists a graph G with $hc(G) = 2$ such that each of the two colors is used an arbitrarily large number of times in a minimum hamiltonian coloring of G. Other graphs with hamiltonian chromatic number 2 can be obtained from H_{k-1} by deleting any or all of the edges $u_iv_i(1 \leq i \leq k)$. The constructions described above for producingclasses of graphs with hamiltonian chromatic number 2 can be altered to produce graphs (indeed, hamiltonian graphs) with larger hamiltonian chromatic numbers. Let k and n be integers with $n \geq 2k \geq 4$ and let F_k be the graph of order n obtained by identifying an edge of K_{n-k+1} and an edge of K_{k+1}. Denote the identified edge by uv. Since $n \geq 2k$, it follows that $n - k + 1 \geq k + 1$. Furthermore, $D(u, v) = n - k$. The coloring c that assigns 1 to every vertex of F_k except v and assigns k to v is a hamiltonian coloring of F_k. Since $|c(u) - c(v)| + D(u, v) = (k - 1) + (n - k) = n - 1$, it follows that c is, in fact, a minimum hamiltonian coloring of F_k and so $hc(F_k) = k$. Of course, $hc(F_k - uv) = k$ as well. This gives us the following result:

Theorem 4.4.1. For every two integers k and n, where $1 \leq k \leq bn/2]$, there exists a hamiltonian graph G of order n with $hc(G) = k$.

Theorem 4.4.1 can be extended however. First, the following lemma will be useful.

Lemma 4.4.1. Let G be a connected graph of order n and H an induced subgraph of order k in G. If $D_H(u, v) \geq D_G(u, v) - (n - k)$ for every two distinct vertices u and v of H, then $hc(H) \leq hc(G)$.

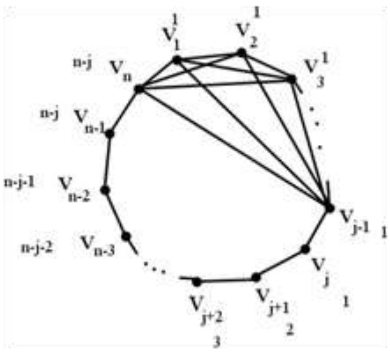
Proof 4.4.1. Let c be a minimum hamiltonian coloring of G and cj the restriction of c to H. Let $u, v \in V(H)$. Since $D_H(u, v) \geq D_G(u, v) - (n - k)$, it follows that

$$|c_j(u) - c_j(v)| + D_H(u, v) \geq |c(u) - c(v)| + D_G(u, v) - (n - k)$$
$$\geq (n - 1) - (n - k) = k - 1.$$

Thus cj is a hamiltonian coloring of H an d so $hc(H) \leq hc(c_j) \leq hc(c) = hc(G)$.

Theorem 4.4.2. Let j and n be integers with $2 \leq j \leq (n + 1)/2$ and $n \geq 6$. Then there is a hamiltonian graph of order n with hamiltonian chromatic number $n - j$.

Proof 4.4.2. Let $G = G(n, j)$ be the graph consisting of the cycle $C_n : v_1, v_2, \cdots, v_n, v_1$ together with all edges joining vertices in $\{v_1, v_2, \cdots, v_{j-1}, v_n\}$. If $j = 2$, then $G = G(n, 2) = C_n$. Since $hc(C_n) = n - 2$, we can assume that $j \geq 3$. Define a coloring $c*$ of V(G) by



(A hamiltonian coloring c∗ of G)

The graph G together with the coloring c. is shown in Figure 4.9. It is straightforward to show that c∗ is a hamiltonian coloring. Thus hc(G) ≤ hc(c∗) = n − j. Next we show that hc(G) ≥ n − j. Let

H = ({vj−1, v j, v j+1, · · · ,vn}) = Cn− j+2. Thus hc(H) = n − j by Theorem 4.3.3. With k = n − j + 2, we see that G and H satisfy the conditions in Lemma 4.4.1. It then follows that n − j = hc(H) ≤ hc(G), completing the proof.□

Combining Theorem 4.4.1 and Theorem 4.4.2, we have the following corollary:

Corollary 4.4.1. For every two integers k and n with 1 ≤ k ≤ n−2, there is a hamiltonian graph of order n with hamiltonian chromatic number k.

We now know that for every integer n ≥ 3, there exists a graph G of order n with a certain specified hamiltonian chromatic number. But how large can the hamiltonian chromatic number of a graph of order n be? In order to answer this question, we present an upper bound for the hamiltonian chromatic number of a graph in terms of its order. We begin with an observation. Let G be a connected graph containing an edge e such that G −e is connected. For every two distinct vertices u and v in G −e, the length of a longest u − v path in G does not exceed the length of a longest u − v path in G − e Thus every hamiltonian coloring of G − e is a hamiltonian coloring of G. This observation yields the following lemma.:

Lemma If e is an edge of a connected graph G such that G − e is connected, then hc(G) ≤ hc(G − e).

Combining Theorem 4.3.3 and Lemma 4.4.2, we have the following theorem:

Theorem . If G is a hamiltonian graph of order n ≥ 3, then hc(G) ≤ n − 2. Definition 4.4.1. The length of a longest cycle in a connected graph is called the circumference of G and is denoted by cir(G).

Theorem . If G is a connected graph of order n ≥ 4 with cir (G) = n − 1, then hc(G) ≤ n − 1.

Proof . Since G is connected and cir (G) = n − 1, it follows that G contains a spanning subgraph H obtained by adding a pendant edge to a cycle of length n − 1. By Lemma 4.2.1, hc(H) = n − 1, and by Lemma 4.4.2, hc(G) ≤ n − 1.

Indeed, by Corollary 4.4.1, every pair k, n of integers with 1 ≤ k ≤ n − 2 can be realized as the hamiltonian chromatic number and the order of some hamiltonian graph. Consequently, this result cannot be improved. Lemma 4.4.2 also provides the following result:

Theorem.If T is a spanning tree of a connected graph G, then hc(G) ≤ hc(T). Definition . The complement G of a graph G is the graph with vertex set V(G) such that two vertices are adjacent in G if and only if they are not adjacent in G.

Lemma .If T is a tree of order at least 4, that is not a star, then T contains a hamiltonian path.

Proof . We proceed by induction on the order n of T. For n = 4, the path P4 of order 4 is the only tree of order 4 that is not a star. SinceP¯4 = P4, the result holds for n = 4. Assume that for every tree of order k − 1 ≥ 4 that is not a star, its complement contains a hamiltonian path. Now let T be a tree of order k that is not a star. Then T contains an end-vertex v such that T − v is not a star. By the induction hypothesis, T − v contains a hamiltonian path, say v1, v2, · · · , vk−1. Since v is an end-vertex of T, it follows that v is adjacent to at most one of v1 and vk−1. Without loss of generality, assume that v1 and v are not adjacent in T. Then v and v1 are adjacent in T and so v, v1, v2, · · · , vk−1 is a hamiltonian path in T .

Theorem . If T is a tree of order n ≤ 2, then hc(T) ≤ (n − 2)2 + 1.

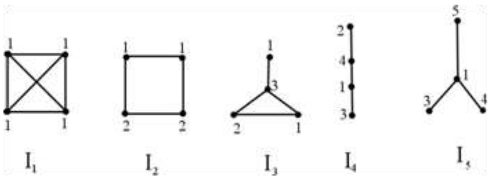
Proof 4.4.5. If T is a star, then by Theorem 4.3.1, hc(T) = (n − 2)2 + 1 and the result holds. So we may assume that T is a tree of order n ≥ 4 that is not a star. By Lemma 4.4.3, the complement T of T contains a hamiltonian path, say v1, v2, · · · ,vn is a hamiltonian path in T . This implies that for each i with 1 ≤ i ≤ n, the vertices vi and vi+1 are nonadjacent in T. Thus D(vi, vi+1) ≥ 2 for all i with 1 ≤ i ≤ n − 1. Define a labeling c of T by c(vi) = (n − 2) + (i − 2)(n − 3) for each i with 1 ≤ i ≤ n. Let 1 ≤ i < j ≤ n. Then |c(vi) − c(vj)| = (j − i)(n − 3). If j = i + 1, then |c(vi) − c(vj)| + D(vi, v j) ≥ (n − 3) + 2 = n − 1.

If j ≥ i + 2, then |c(vi) − c(vj)| + D(vi, v j) ≥ 2(n − 3) + 1 = 2n − 5 ≥ n − 1 for n ≥ 4. Thus c is a hamiltonian coloring of T. Therefore, hc(T) ≤ hc(c) = c(vn) = (n − 2)2 < (n − 2)2 + 1, as desired.

As a consequence of Theorems 4.4.5 and 4.4.6, we obtain a sharp upper bound for the hamiltonian chromatic number of a nontrivial connected graph in terms of its order.

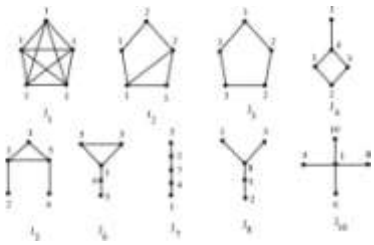
Corollary 4.4.2. If G is a nontrivial connected graph of order n, then hc(G) ≤ (n−2)2+1.

The preceding results suggest defining the following set and parameter for each integer n ≥ 2, HC(n) = {k : there exists a graph G of order n with hc(G) = k}. Therefore, min{HC(n)} = 1 and max{HC(n)} = (n − 2)2 + 1. Also, hc(n) = max{k : p ∈ HC(n) for all



(Graphs Ii of order 4 with hc(Ii) = i (1 ≤ i ≤ 5))

1 ≤ p ≤ k}. By Theorem 4.4.4, Theorem 4.3.1, Corollaries 4.4.1, and 4.4.2, it follows thatn − 1 ≤ hc(n) ≤ (n − 2)2 + 1. That HC(4) = {1, 2, 3, 4, 5} and HC(5) ={1, 2, · · · , 10} − {9} is illustrated in Figures 4.10 and 4.11 Consequently, hc(4) = 5 andhc(5) = 8. Among the many unsolved problems is to determine those integers n ≥ 2 for which n ∈ HC(n).



(Graphs Ji of order 5 with hc(Ji) = i (1 ≤ i ≤ 10, i ≠ 9))

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