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# A Study on $\alpha(\beta)-\rho-$ Open (Closed) Sets and Related Concepts in Topology

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Generalized continuity connectedness and compactness in a topological spaces have been extensively studied by general topologists.

Any function $f: x \longrightarrow$y induces a multi function $f^{-1}of: x \rightarrow \delta(x)$. It is also induces another multifunction $fof^{-1}: y \rightarrow \delta(y)$provided $f$ is subjective.

Priyadarshini.M, Selvi.R introduced the concept of Generalized$-\rho-$continuity,$\rho-$ connectedness and $\rho-$ compactness where $\rho\epsilon\{L, M, R, S\}$[7].

In this work, $\alpha(\beta)-\rho-O$ open (closed) sets is introduced and its properties are investigated.

In topology and related branches of mathematics a connected space is a topological spaces that cannot be represented as the union of two disjoint non empty open subsets. Connectedness is one of the principal topological properties that are used to distinguish topological spaces. In this work we introduce $\alpha$ - $\rho$ -connected spaces.

A topological space X is said to be $\alpha$ - $\rho$ -connected if X cannot be written as the disjoint union of two non-empty $\alpha$ - $\rho$ -open or $\alpha$ - $\rho$ -closed sets in X and we introduce $\beta$ - $\rho$ -connected spaces. A topological space X is said to be $\beta$ - $\rho$ -connected if X cannot be written as the disjoint union of two non- empty $\beta$ - $\rho$ -open or $\beta$ - $\rho$ -closed sets in X.

S.Pious Missier, A.Robert [10] introduced a new class of $\alpha$ -open sets in 2014. We introduced a new class of $\alpha$ - $\rho$ -open(closed) sets.

In this work $\alpha$-L-compact, $\alpha$-R-compact, $\alpha$-L-locally compact, $\alpha$-R-locally compact, sequentially $\alpha$-L-compact, sequentially $\alpha$-R-compact, countably $\alpha$-L-compact, countably $\alpha$-Rcompact are defined and their properties are investigated and

Andrijevic D [1] introduced a new class of $\beta$-open sets in 1986, and in this work we introduced a new class of $\beta$-$\rho$-open(closed) sets.In this paper $\beta$-L-compact, $\beta$-R-compact, $\beta$-L-locally compact, $\beta$-R-locally compact, sequentially $\beta$-L-compact, sequentially $\beta$-R-compact, countably $\beta$-L-compact, countably $\beta$-R-compact are defined and their properties are investigated.

## $\beta-\rho$– Compact Space

**Definition**

(i) A collection $\{U_\alpha\}_{\alpha\in\Delta}$ of $\beta$-L-open sets in X is saidto be $\beta$-L-open cover of X if X= $\cup_{\alpha\in\Delta} U_\alpha$ .(ii)A collection $\{U_\alpha\}_{\alpha\in\Delta}$ of $\beta$-R-open sets in X is said to be $\beta$-R-open cover of X if X= $\cup_{\alpha\in\Delta} U_\alpha$ .

**Definition**

(i) A topological space $(X,\tau)$ is said to be $\beta$-L-compact if every $\beta$-L-open covering of X contains finite sub collection that also cover X.A subset A of X is said to be $\beta$-L-compact if every covering ofA by $\beta$-L-open sets in X contains a finite subcover. (ii) A topological space $(Y,\sigma)$ is said to be $\beta$-R-compact if every $\beta$-R-open covering of Y contains finite sub collection that also cover Y. A subset B of Y is said to be $\beta$-R-compact if every covering of B by $\beta$-R-open sets in Ycontains a finite subcover.

**Theorem**

A topological space $(X,\tau)$ is

1) $\beta$-L-compact $\Rightarrow$ compact
2) Any finite topological space is $\beta$-L-compact.

**Proof:**

1) Let $\{A_\alpha\}_{\alpha\in\Omega}$ be an open cover for X. Then each $A_\alpha$ is $\beta$-L- open.Since X is $\beta$-L-compact, this open cover has a finite subcover. Therefore $(X,\tau)$ is compact.
2) Obvious since every $\beta$-L-open cover is finite.

**Example**

Let $(X,\tau)$ be an infinite indiscrete topological space. In this space all subsets are $\beta$-L-open. Obviously it is compact. But {x}x$\in$X is a $\beta$-L-open cover which has no finite subcover. So it is not $\beta$-L-compact. Hence compactness need not imply $\beta$-L-compactness.

**Theorem**

A $\beta$-M-closed subset of $\beta$-L-compact space is $\beta$–L-compact.

**Proof**

Let A be a $\beta$-M-closed subset of a $\beta$-L-compact space $(X,\tau)$ and $\{U_\alpha\}_{\alpha\in\Delta}$ be a $\beta$–L-open cover for A , then $\{\{U_\alpha\}_{\alpha\in\Delta}, \{X-A\}\}$ is a $\beta$-L-open cover for X. Since X is $\beta$-L-compact, there exists $\alpha_1,\alpha_2....\alpha_n\in\Delta$ such that

$$X = U\alpha_1 \cup U\alpha_2 ...... \cup U\alpha_n \cup (X-A)$$

Therefore $A \subseteq U\alpha_1 \cup U\alpha_2 ...... \cup U\alpha_n$ which proves A is $\beta$-L-compact.

**Remark**

The converse of the above theorem need not be true as seen in the following example (**8.1**.7).

**Example**

Let X = {a, b, c} and Y = {1, 2, 3}. Let f: (X,$\tau$) $\rightarrow$Y defined by f(a)=1, f(b)=2, f(c)=3. Let X={a,b,c} $\tau$ ={$\phi$,{a},X}-open set, closed set-{ $\phi$, X, {b, c}}. Here $\beta$-L-Open (X) = {$\phi$, X, {a} {a, b}, {a, c}} is $\beta$-L-compact, A={a, c} is $\beta$-L-compact but not $\beta$-M-closed.

**Theorem**

A topological space $(X,\tau)$ is $\beta$-L-compact if and only if for every collection $\tau$ of $\beta$-M-closed sets in X having finite intersectionproperty, $\bigcap_{c\in\tau} C$ of all elements of $\tau$ is non empty. **PROOF:**

Let $(X,\tau)$ be $\beta$-L-compact and $\tau$ be a collection of $\beta$-M-closed sets with finite intersection property. Suppose $\bigcap_{c\in\tau} C \;=\phi$ then $\bigcup_{c\in C}(X-C)=X$. Therefore $\{X-C\}_{c\in C}$ is a $\beta$-L-open cover for X. Then there exists $C_1$, C2,......Cn$\in\tau$ such that $\bigcup_{i=1}^{n}(X-C_i)$ =X Therefore $\bigcap_{i=1}^{n} C_i=\phi$ which is a contradiction. Therefore $\bigcap_{c\in\tau} C\neq\phi$ Conversely assume the hypothesis given in the statement. To prove X is $\beta$-L-compact. Let {U $\alpha$ } $\alpha\in\Delta$ be a $\beta$-L-open cover for X.

Then $\bigcup_{\alpha\in\Delta} U_\alpha = X \Rightarrow \bigcap_{\alpha\in\Delta}(X-U_\alpha)=\phi$ By hypothesis $\alpha_1,\alpha_2....\alpha_n$, there exists such that $\bigcap_{i=1}^{n}(X-U_{\alpha_i})=\phi$. Therefore $\bigcup_{i=1}^{n} U_{\alpha_i}=X$. Therefore X is $\beta$-L-compact.

**Corollary**

Let $(X,\tau)$ be a $\beta$-L-compact space and let $C_1\supseteq C_2\supseteq......\supseteq C_n\supseteq C_{n+1}....$ be a nested sequence of non empty $\beta$-M-closed sets in X. then $\bigcap_{n\in Z^+} C_n$ is non empty.

**Proof**

Obviously $\{C_n\}_{n\in Z^+}$ finite intersection property. By theorem (7.8) $\bigcap_{n\in Z^+} C_n$ is non empty.

**Theorem**

Let $(X,\tau)$, (Y,$\sigma$ ) be two topological space and f: (X,$\tau$) $\rightarrow$(Y,$\sigma$ ) be a bijection then

1) f is $\beta$- continous and X is $\beta$–L-compact $\Rightarrow$Y is compact.
2) f is $\beta$–L-irresolute and X is $\beta$- L-compact $\Rightarrow$Y is $\beta$-L- compact .
3) f is continous and X is $\beta$-L-compact$\Rightarrow$Y is compact.
4) f is strongly irresolute and X is compact $\Rightarrow$Y is $\beta$- L-compact.
5) f is $\beta$ –L-open and Y is $\beta$- L-compact $\Rightarrow$X is compact.
6) f is open and Y is $\beta$-L- compact $\Rightarrow$X is compact.
7) f is pre- R-resolute and Y is $\beta$-R-compact $\Rightarrow$ X is $\beta$- R-compact.

**Proof:**

1) Let {U $\alpha$ } $\alpha\in\Delta$ be a open cover for Y.

Therefore $Y=\cup U_\alpha$. Therefore $X=f^{-1}(Y)=\cup f^{-1}(U_\alpha)$ .Then {$f^{-1}$(U $\alpha$ )} $\alpha\in\Delta$ is a $\beta$-L- open cover for X . Since X is $\beta$-L- compact, there exists $\alpha_1,\alpha_2....\alpha_n$ suchthat $X=\cup f^{-1}(U_{\alpha_i})$.

Therefore $Y=f(X)=\cup(U_{\alpha_i})$ .ThereforeY is compact. Proof of (2) to (4) are similar to the above. 2) Let {U $\alpha$ } $\alpha\in\Delta$ be a open cover for X. then {f(U $\alpha$ )} is a $\beta$-L-open cover for Y. Since Y is $\beta$-L-compact ,there exists $\alpha_1,\alpha_2....\alpha_n$ such that $Y=\cup f(U_\alpha)$

Therefore $X=f^{-1}(Y)=\cup_{\alpha\in\Delta}(U_\alpha)$.

Therefore X is compact. Proof of (6) and (7) are similar.

**Remark**

From (3) and (6) it follows that "$\beta$-L-compactness" is a $\beta$-L- topological property.

**Theorem: (Generalisation of Extreme Value Theorem)**

Let f: X $\rightarrow$ Y be $\beta$-L-continuous where Y is an ordered set in the ordered topology. If X is $\beta$-L-compact then there exists c and d in X such thatf(c) $\leq$ f(x) $\leq$ f(d) for every x $\in$ X.

**Proof**

We know that $\beta$-L-continuous image of a $\beta$-L-compact space is compact By Theorem (**8.1**.10). Therefore A=f(X) is compact. Suppose A has no largest element then $\{(-\infty, a) / a \in A\}$ form an open cover for A and it has a finite subcover .
Therefore $A \subseteq (-\infty, a_1) \cup (-\infty, a_2) \cup ..... \cup (-\infty, a_n)$. Let $a = \max_i a_i$.

Then $A \subseteq (-\infty, a)$ which is a contradiction to the fact that $a \in A$

Therefore A has a largest element M. Similarly it can be proved that it has the smallest element m. Therefore $\exists$ c and d in X $\ni$ f(c) = m, f(d) = M and f(c) $\leq$ f(x) $\leq$ f(d) $\forall$ x $\in$ X.

**Countably $\beta - \rho$-Compact Space**

**Definition**

(i) A subset A of a topological space $(X, \tau)$ is said to be countably $\beta$-L-compact, if every countable $\beta$-L-open covering of A has a finite subcover.

(ii) A subset B of a topological space $(Y, \sigma)$ is said to be countably $\beta$-R-compact, if every countable $\beta$-R-open covering of B has a finite subcover.

**Example**

Let $(X, \tau)$ be a countably infinite indiscrete topological space.

In this space {{x}/ x $\in$ X} is a countable $\beta$-L-open cover which has no finite subcover. Therefore it is not countably $\beta$-L-compact.

**Remark**

(i) Every $\beta$-L-compact space is countably $\beta$-L-compact. It is obvious from the definition.

(ii) Every countably $\beta$-L compact space is countably compact. It follows since open sets are $\beta$-Lopen.

**Theorem**

In a countably $\beta$-L-compact topological space, every infinite subset has a $\beta$-L-limit point.

**Proof**

Let $(X, \tau)$ be countably $\beta$-L-compact space. Suppose that there exists an infinite subset A which has no $\beta$-L-limit point. Let $B = \{a_n / n \in N\}$ be a countable subset of A.

Since B has no $\beta$-L-limit point of B, there exists a $\beta$-L-neighbourhood $U_n$ of $a_n$ such that $B \cap U_n = \{a_n\}$. Now $\{U_n\}$ is a $\beta$-L-open cover for B .Since $B^c$ is $\beta$-L-open, $\{B^c, \{U_n\}_{n \in Z^+}\}$ is a countable $\beta$-L-open cover for X. But it has no finite sub cover, which is a contradicition, since X is countably $\beta$-L-compact. Therefore every infinite subset of X has a $\beta$-L-limit point.

**Corollary**

In a $\beta$-L-compact topological space every infinite subset has a $\beta$-L-limit point.

**Proof**

It follows from the theorem (**8.2**.4), since every $\beta$-L-compact space is countably $\beta$-L-compact.

**Theorem**

A $\beta$-M-closed subset of countably $\beta$-L-compact space is countably $\beta$-L-compact.

**Proof**

Let X be a $\beta$-L-compact space and B be a $\beta$-M-closed subsets of X. Let $\{A_i / i = 1,2,3,...\infty\}$ be a countable $\beta$-L-open cover for B. Then $\{\{A_i\}, X - B\}$ where $i = 1,2,3,....\infty$ is a $\beta$-L-open cover for X. Since Xis countably $\beta$-L-compact, there exists $i_1, i_2, i_3.....i_n \ni (X - B) \bigcup_{k=1}^{n} A_{ik} = X$. Therefore $B = \bigcup_{k=1}^{n} A_{ik}$ and this implies B is countably $\beta$-L-compact.

**Definition**

In a topological space $(X,\tau)$ a point $x \in X$ is said to be a $\beta$-L-isolated point of A if there exists a $\beta$-L-open set containing x which contains no point of A other than x.

**Theorem**

A topological space $(X,\tau)$ is countably $\beta$-L-compact if and only if for every countable collection $\tau$ of $\beta$-L-closed sets in X having finite intersection property, $\bigcap_{c\in C} C$ of all elements of $\tau$ is non empty.

**Proof:** It is similar to the proof of theorem (**8.2**.8).

**Corollary**

X is countably $\beta$-L-compact if and only if every nested sequence of $\beta$-M-closed non empty sets $C1 \supset C2 \supset .....$.has a non empty intersection.

**Proof:**

Obviously $\{C_n\}_{n\in Z^+}$ has finite intersection property. By theorem (**8.2**.8) $\bigcap_{n\in Z^+} C_n$ is non empty.

**Sequentially $\beta-\rho$-Compact Space**

**Definition**

(i) A subset A of a topological space $(X,\tau)$ is said to be sequentially $\beta$-L-compact if every sequence in A contains a subsequence which $\beta$-L-converges to some point in A.

(ii) A subset B of a topological space $(Y,\sigma)$ is said to be sequentially $\beta$-R-compact if every sequence in B contains a subsequence which $\beta$-R-converges to some point in B.

**Theorem**

Any finite topological space is sequentially $\beta$-L-compact.

**Proof**

Let $(X,\tau)$ be a finite topological space and $\{x_n\}$ be a sequence in X. In this sequence except finitely many terms all other terms are equal. Hence we get a constant subsequence which $\beta$-L-converges to the same point .

**Theorem**

Any infinite indiscrete topological space is not sequentially $\beta$-L-compact.

**Proof**

Let $(X,\tau)$ be infinite indiscrete topological space and $\{x_n\}$ be a sequence in X. Let $x \in X$ be arbitrary. Then U={x} is $\beta$-L- open and it contains no point of the sequence except x. Therefore $\{x_n\}$ has no subsequence which $\beta$-L-converges to x. Since x is arbitrary, X is not sequentially $\beta$-L-compact.

**Theorem**

A finite subset A of a topological space $(X,\tau)$ is sequentially $\beta$-L-compact.

**Proof**

Let $\{x_n\}$ be an arbitrary sequence in X. Since A is finite, at least one element of the sequence say $x_0$ must be repeated infinite number of times. So the constant subsequence $x_0, x_0$ ….. must $\beta$-L-converges to $x_0$.

**Remark**

Sequentially $\beta$-L-compactness implies sequentially compactness, since all open sets are $\beta$-L-open. But the inverse implication is not true as seen from (9.6).

**Example**

Let $(X,\tau)$ be an infinite indiscrete space is sequentially compact but not sequentially $\beta$-L-compact.

**Theorem**

Every sequentially $\beta$-L-compact space is countably $\beta$-compact.

**Proof**

Let $(X,\tau)$ be sequentially $\beta$-L-compact. Suppose X is not countably $\beta$-L-compact. Then there exists countable $\beta$-open cover $\{U_n\}_{n\in Z^+}$ whichhas no finite sub cover. Then $X = \bigcup_{n\in Z^+} U_n$. Choose

$$X_1 \in U_1, X_2 \in U_2 - U_1, X_3 \in U_3 - \bigcup_{i=1,2} U_i \ldots\ldots X_n \in U_n - \bigcup_{i=1}^{n} U_i$$

. This is possible since {Un} has no finite sub cover. Now $\{x_n\}$ is a sequence in X. Let $x \in X$ be arbitrary. then $x \in$ Uk for some K .By our choice of $\{x_n\}$, xi $\notin$ Uk for all $i \geq k$. Hence there is no subsequence of $\{x_n\}$ which can $\beta$-L-converge to x. Since x is arbitrary the sequence $\{x_n\}$ has no $\beta$-L-convergent subsequence which is a contradiction. Therefore X is countably $\beta$-L-compact.

**Theorem:**

Let f: $(X, \tau) \to (Y, \sigma)$ be a bijection, then

1) f is $\beta$-R-resolute and Y is sequentially $\beta$-R-compact $\Rightarrow$ X is sequentially $\beta$ -R-compact.
2) f is $\beta$-L-irresolute and X is sequentially $\beta$-compact $\Rightarrow$ Y is sequentially $\beta$-L-compact.
3) f is continuous and X is sequentially $\beta$-L-compact $\Rightarrow$ Y is sequentially $\beta$-L-compact.
4) f is strongly $\beta$-L-continous and X is sequentially $\beta$-L-compact $\Rightarrow$ Y is sequentially $\beta$-L-compact.

**Proof**

1) Let $\{x_n\}$ be a sequence in X .Then $\{f(x_{nk})\}$ is a sequence in Y. It has a $\beta$–R-convergent subsequence $\{f(x_{nk})\}$ such that $\{f(x_{nk})\} \xrightarrow{pre} y_0$ in Y. Then there exists $x_0 \in$ X such that $f(x_0) = y_0$. Let U be $\beta$-R -open set containing x0 then f(U) is a $\beta$-R-open set containing y0 .Then there exists N such that f $\in$ f(U) for all k $\geq$ N.

   Therefore $f^{-1} \circ f(x_{nk}) \in f^{-1} \circ f(U)$.

   Therefore $x_{nk} \in U$ for all $k \geq N$.

   This proves that X is sequentially $\beta$-R-compact. Proof for (2) to (4) is similar to the above.

**Remark**

From theorem (**8.3**.8), (1) and (2) it follows that "Sequentially compactness" is a $\beta$-$\rho$-topological property.

## $\beta - \rho$ -Locally Compact Space

**Definition**

A topological space $(X,\tau)$ is said to be $\beta$-L-locally compact if every point of X is contained in a $\beta$-L-neighbourhood whose $\beta$-L-closure is $\beta$-L-compact.

**Theorem:**

Any $\beta$-L-compact space is $\beta$-L-locally compact.

**Proof**

Let $(X,\tau)$ be $\beta$-L-compact, Let $x \in X$ then X is $\beta$-L-neighbourhood of x and $\beta$ cl(X) = X which is $\beta$-L-compact .

**Remark**

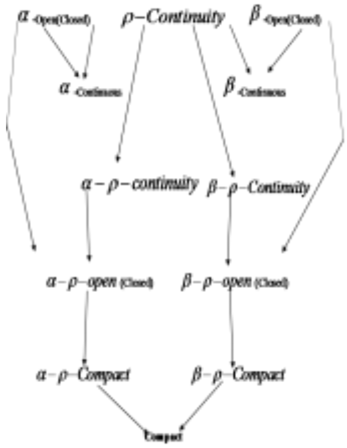
The converse need not be true as seen in the following example

**Example**

Let $(X,\tau)$ be an infinite indiscrete topological space. It is not $\beta$-L-compact. But for every $x \in X$, {x} is a $\beta$-L-neighbourhood and $\{\overline{x}\} = \{x\}$ is $\beta$-L-compact.

Therefore it is $\beta$-L-locally compact.

**The Above Discussions Give the Following Implication Diagram**




**Conclusion**

The authors introduced $\alpha$-L-open sets, $\alpha$-M-closed sets, $\alpha$-R-open sets and $\alpha$-S-closed sets and established their relationships with some generalized sets in a topological spaces. Connected spaces constitute the most important classes of topological spaces.

In this work we introduce the concept "$\alpha$ - $\rho$ -connected" in a topological space.

And also $\beta$-L-open sets, $\beta$-M-closed sets, $\beta$-R-open sets and $\beta$-S-closed sets and established their relationships with some generalized sets in topological spaces. Connected spaces constitute the most important classes of topological spaces.

We introduce the concept "$\beta$ - $\rho$ -connected" in a topological space.

Also Compact spaces constitute the most important classes of topological spaces. In this work we introduce the concept "$\alpha$ - $\rho$ -compact" and"$\beta$ - $\rho$ -compact" in a topological space.

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