A Study on Labeling of Standard Graphs

OPEN ACCESS

Volume: 10

Special Issue: 1

Month: August

Year: 2022

P-ISSN: 2321-788X

E-ISSN: 2582-0397

Received: 28.06.2022

Accepted: 20.08.2022

Published: 30.08.2022

Citation:
Sumathi, S., and L. Jethruth Emelda Mary. "A Study on Labeling of Standard Graphs." *Shanlax International Journal of Arts, Science and Humanities*, vol. 10, no. S1, 2022, pp. 121–41.

DOI:
https://doi.org/10.34293/sijash.v10iS1.5262

S. Sumathi
M.Phil. Scholar in Mathematics (PT), Reg. Number: 197215EP157
Mother Teresa Women's University, Kodaikanal

Dr. L. Jethruth Emelda Mary, M.Sc., M.Phil., Ph.D.,
Assistant Professor of Mathematics
St. Joseph's College of Arts & Science (Autonomous), Cuddalore

This part introduces the fundamental terminology that will be used throughout the dissertation and defines the fundamental ideas of graph theory.

A graph G is made up of vertices, which are a finite non-empty collection of objects, and edges, which are a set E of 2-element subsets of vertices. The sets V and E are, respectively, the vertex set and edges set of G. Thus, two sets V and E make up a graph, or G. This is why some individuals write G = (V, E).

- To stress that these are the vertex and edge set of a certain graph G, it is helpful to write V (G) and E (G) rather than just V and E.
- The standard symbol for a graph is G, although we may also use F, H, G', G'' and G1, G2, etc.
- Vertices are often referred to as points or nodes, and edges are occasionally referred to as lines.
- Parallel edges are any number of edges that connect the same two different vertices.
- A loop is an edge that is represented by an unordered pair with non-distinct members.

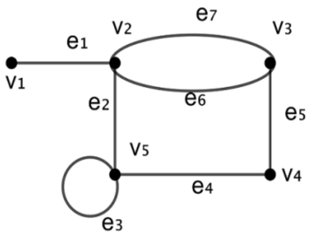
List of Symbols

n	Order of the graph
m	Size of the graph
V (G)	Vertex set of a graph G
E (G)	Edge set of a graph G
$\cong$	Isomorphic
$\subseteq$	Subset
$\langle S \rangle$	Induced sub-graph induced by
$\overline{G}$	Complement of a graph G
$\deg V$	Degree of a vertex V
δ (G)	Minimum degree of a graph G
Δ (G)	Maximum degree of a graph G

K_n	Complete graph on n vertices
$K_{m, n}$	complete bipartite graph on m + n vertices
$K_{1, n}$	Star graph
diam (G)	diameter of a graph G
	rad (G) radius of a graph G
V/G	V restricted to G
N (V)	Open neighborhood of a vertex V
N [V]	Closed neighborhood of a vertex V

Definition: 1.1 Graph

A graph G = (V(G), E(G)) is made up of two finite sets: V(G), the graph's vertex set, which is frequently abbreviated as simply V, and E(G), the graph's edges set, which is occasionally abbreviated as just E and may or may not be an empty set of items called edges.



This is a graph G, with five vertices and seven edges.

V (G) = {V_1, V_2, V_3, V_4, V_5,}

E (G) = {e_1, e_2, e_3, e_4, e_5, e_6, e_7,}

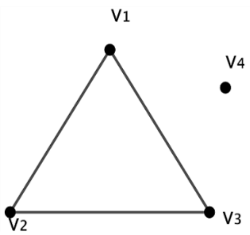
Definition: 1.2 Empty Graph

An empty graph is a graph with no edges.

This is an empty graph with only two vertices and no edges.

Definition: 1.3 Isolated

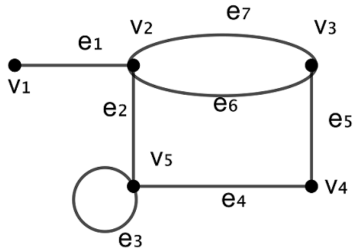
A vertex of G which is not an endpoint of any edge is called isolated.



A graph G with isolated vertex v_4

Definition: 1.4 Parallel

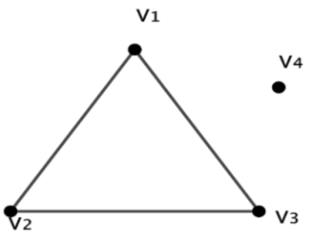
Let G be a graph. If two edges of G have the same end vertices, then these edges are called parallel.



The edges e_6 and e_7 of the graph are parallel.

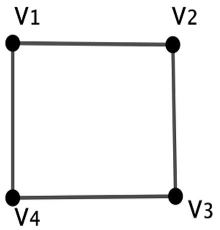
Definition: 1.5 Adjacent

Two vertices which are joined by an edge are said to be adjacent (or) neighbors. In the graph v_2 and v_3 are adjacent but v_1 and v_4 are not adjacent.



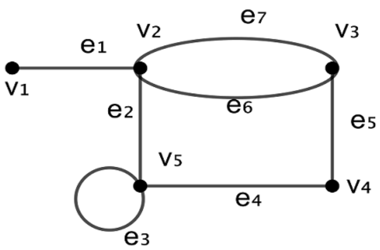
Definition: 1.6 Simple Graph

A graph is called simple if it has no loops and no parallel edges.



Definition: 1.7 Multi Graph

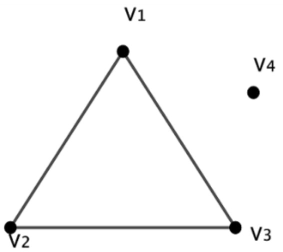
A graph which is not simple is called a multi-graph.



In the graph, G is a multi graph.

Definition: 1.8 Neighbourhood Set

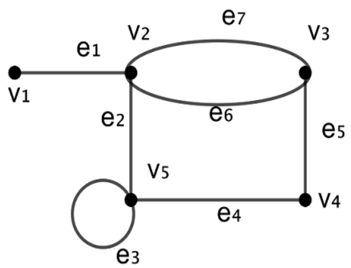
The set of all neighbors of a fixed v of G is called the neighborhood set of v and is denoted by N (v).



In the graph of the neighborhood set N (v_1) of v_1 is {v_2, v_3}.

Definition: 1.9 Loop

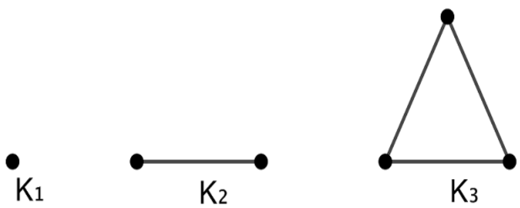
It is possible to have a vertex v joined to by an edge; such an edge is called as a loop.



In the graph the vertex v_5 has the loop.

Definition: 1.10 Complete Graph

A complete graph is a simple graph in which each pair of distinct vertices is joined by an edge. It is denoted by K_n.

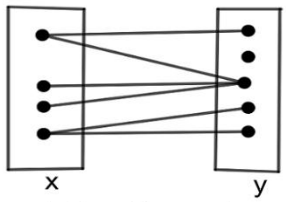


These are complete graph with one, two, and three vertices.

Definition: 1.11 Bipartite Graph

If a graph G has a single vertex set and no edges, it is considered trivial.

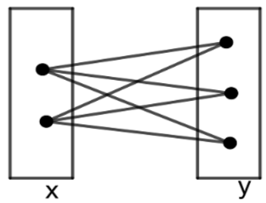
If a graph's vertex set can be divided into two nonempty subsets, X and Y, with each edge of G having one end in each subset, then the graph is bipartite. A bipartition of the bipartite graph is the pair (X, Y). G stands for the bipartite graph G with bipartition (X, Y).



A bipartite graph

Definition: 1.12 Complete Bipartite Graph

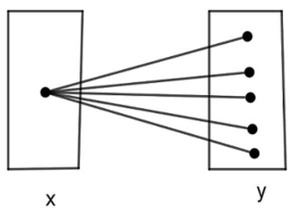
A simple bipartite $G(X,Y)$ is complete if each vertex of X is adjacent to all the other vertices of Y. If $G(X,Y)$ is complete with X| = p and |Y| = q, then $G(X,Y)$ is denoted by $K_{p,q}$.



The graph $K_{2,3}$

Definition: 1.13 Star Graph

A complete bipartite graph of the form $K_{1,\ q}$ is called a star.

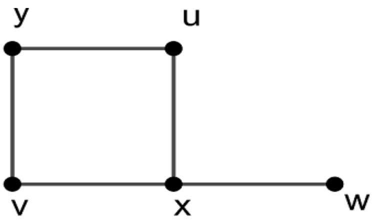


The graph $K_{2,3}$

Definition: 1.14 Vertex Independent Sets

A subset S of the vertex set V of a graph G is called independent if no two vertices of S are adjacent in G. $S \subseteq V$ is a maximum independent set of G if G has no independent set S' with $|S'| > |S|$. A maximum independent set that is not a proper subset of another independent set of G.

For example, in the graph of figure $\{u, v, w\}$ is a maximum independent set and $\{x, y\}$ is maximal of that is not maximum.

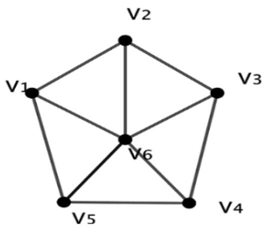


{u, v, w} → Maximum independent set.
{x, y} → Maximum independent set.

Definition: 1.15 Covering

A subset k of V is called a covering of G if every edge of G is incident with at least one vertex of k. A covering k is minimum if there is no covering k' of G such that |k'| □ |k| it is minimal if there is no covering k_1 of G such that k_1 is a proper subset of k.

Example



$\{v_6\}$ is the minimum covering of the figure.

In the graph w_5 of figure $\{v_1, v_2, v_3, v_4, v_5\}$ is a covering of w_5 and $\{v_1, v_3, v_4, v_6,\}$ is a minimal covering. Also the set $\{x, y\}$ is a minimum covering of the graph of figure.

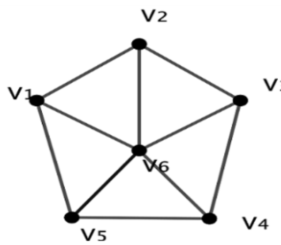
Definition: 1.16 Edge Independent Set

A subset M of the edge set E of a loop less graph G is called independent if no two edges of M are adjacent in G.

6. A matching in G is a set of independent edge.
7. An edge covering of G is a subset L of E such that every vertex of G is incident to some edge of L. Hence an edge covering of G exists if $\delta > 0$.
8. A matching M of G is maximum if G has no matching M' with $|M'| > |M|$. M is maximum strictly containing M. ά (G) is the cardinality of a maximum matching and β'(G) is the size of a minimum edge covering of G.
9. A set of vertices of G is said to be saturated by a matching M of G or M-saturated if every vertex of S is incident to some edge of M.A vertex v of G is M-saturated if {v} is M-saturated. V isM-unsaturated if it is not M- saturated.

Definition: 1.17 Augmenting Path

A path in G is said to be M-augmenting if its end vertices are M-saturated and its edges alternate between E/M and M. A path in G whose edges alternate between E/M and M is known as an M-alternating path.



Example

In the graph G of the above figure, M_1 = $\{v_1v_2, v_3v_4, v_5v_6\}$ and,

$M_2 = \{v_1v_2, v_3v_6, v_4v_5\}$ and,

$M_3 = \{v_3v_4, v_5v_6\}$ are matching of G. The path $v_2v_3v_4v_6v_5v_1$ is an $M_3 -$ augmenting path in G.

Definition: 1.18 Matching

A matching of a graph G is a set of independent edges of G.

If $e = uv$ is an edges of a matching M of G, the end vertices u and v of e are said to be matched by M.

If M_1 and M_2 are matching of G, the edge sub-graph defined by $M_1 \Delta M_2$, the symmetric difference of M_1 and M_2 is a sub-graph H of G whose components are paths or even cycles of G in which the edges alternative between M_1 and M_2

A matching of a graph G is a set of independent edges of G. If $e = uv$ is an edge of a matching M of G, the end vertices u and v are said to be matched by M.

Definition: 1.19 Perfect Matching

A matching M is called a perfect matching if every point of G is M-saturated M is called a maximum matching if there is no matching M' in G such $|M'| \leq |M|$

Example

Consider the graph G_1 given in figure, $M_1 = \{v_1v_2, v_6v_3, v_5v_4\}$ is a perfect matching in G_1. Also $M_2 = \{v_1v_3, v_6v_5\}$ is matching in G_1. However M_2 is not a perfect matching since the vertices v_2 and v_4 are not M_2 - saturated.

For the graph G_2 given in figure $M = \{v_1v_2, v_8v_4\}$ is a maximum matching but it is not a perfect matching.

For the G_1 given in a figure $P_1 = \{v_6, v_5, v_4, v_3\}$ is an M_1 – alternating path also $P_2 = \{v_2, v_1, v_3, v_6, v_5, v_4\}$ is an M_2 - alternating path.

Definition: 1.19 Perfect Matching

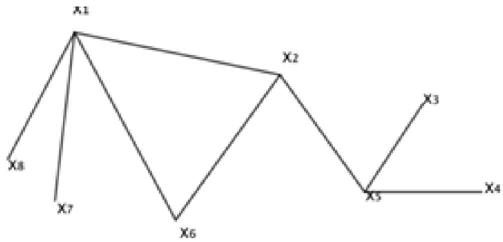
Spanning sub-graph of graph G is a graph G factor. A G factor that is k- regular is referred to as a k- factor. Thus, the first factor of G is a matching that completely fills all of G's vertices, making it the first factor of G's perfect matching.

For example, in the wheel (fig 1) $M = \{v_1v_2, v_4v_6\}$ is a maximal matching; $\{v_1v_5, v_2v_3, v_4v_6\}$ is a maximum matching and a minimum edge covering the vertices v_1, v_2, v_4 and v_6 are M-saturated whereas v_3 and v_5 are M-unsaturated.

Matching in General Graph

Definition: 1.20

A matching is a subgraph of a given graph G = (V, E) where every node has degree 1, in specifically, the matching is made up of edges that do not share any nodes.



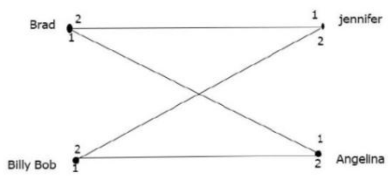
In this graph $x_1 - x_6, x_2 - x_5$ is matching of size two. But there is a large matching normally $x_1 - x_8, x_2 - x_6, x_4 - x_5$ is a matching of size three. Can there be a larger matching? Well, that would mean that every node is paired. But of x_7 and x_8 can only paired with x_1 and x_1 can only be paired with one other node in a matching.

Will you Marry me?

Instead, we'll discuss a different version of the matching issue that is widely utilized in practice and does have an element solution.

Every node in this variant of the issue has a preference order for the potential matches.

The preference need not be symmetric, for instance: Brad has notes from Angelina, so it's possible that Angelina also loves Brad more than Billy Bob, although Billy Bob truly likes Angelina. Jennifer really likes Brad.



If we were to combine Brad with Jennifer in the previous image and Billy Bob with Angelina, things would become extremely complicated! It should go without saying that Brad and Angelina will probably soon start staying up late to work on schoolwork together.

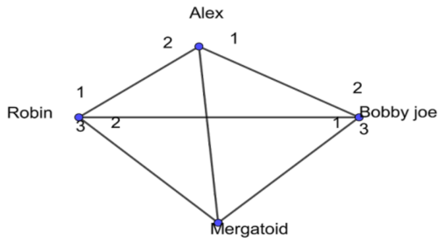
Brad and Angelina prefer each other than their partners in the matching, which is the biggest issue. In this case, we would say that Brad and Angelina make up a rogue pair. To be more specific, we would state that if x and y are excellent for their mates in M, then x and y are a rogue couple for M.

It should go without saying that rogue preferences do not alter over time. Since preferences are established from the beginning and never alter, we are not simulating a scenario in which you become weary of your partner or desire to "play the field."

Finding a steady, ideal fit is our first objective. In this example, a stable perfect pairing could be Brad and Angelina, and Billy Bob and Jen. Billy Bob and Jen might not be happy together, but no rogue couple is possible, making this a stable pairing. This is because even though Jen and Billy Bob aren't happy together, no one else will be able to make them one with either people on Earth or, more logically, suppose you put these four on a desert island.

It is not always obvious that there is a stable match for every number of participants and set of preferences. In actuality, the response is a sort of compliment. There is an example where there is no stable matching if you allow boys to prefer guys and girls to prefer girls. However, the peculiar thing is that you can always discover a steady matching in the unique situation where guys exclusively get paired with girls.

In a moment, we'll demonstrate how to identify such a stable match, but first, let's look at an example for unisex people where a stable match is not achievable. The plan is to set up a love triangle with a fourth person who serves as each individual's last option.



Turns out Mergatoid's preference don't even matter. Let's see why there is no stable matching.

Theorem: 1.21

There is no stable matching.

Proof

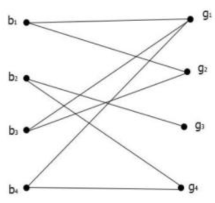
This will be demonstrated through contradiction. Assume there is stable matching in order to create contradiction. If Robin is matched to Alex, then the other pair must be Bobby Joe linked with Mergatoid, but since there is a rogue couple, there cannot be a stable matching. Then there are two numbers of the love triangle that are matched without loosing of generality.

There is little surprise in this theorem. Finding a steady match can be challenging. It is amazing that matching can always be accomplished in a bipartite graph when it is a challenging task.

Where boys are only allowed to pair with girls and vice versa

A matching of a graph G = (V, E) is perfect if it has v/2 edges there is no perfect matching for the previous graph. Matching problems often arise in the context of the bipartite graph.

Example



In the above graph, a perfect matching is normally b_1-g_2, b_2-g_3, b_3-g_1, and b_4-g_4.

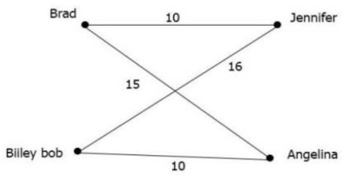
In many applications, not all matching is equally desirable for example, may be b_1 and g_1 like each other a lot more than b_1 and g_2 often, we can represent the desirability of a matching with a weight on the edge for example b_1 and g_2 get weight 5 while b_1 and g_1 get weight 10

The goal is to find the perfect matching with minimum weight.

Definition: 1.22

The total of the weights on M's edges makes up the weight of matching M. With minimum weight, the min-weight for G is the ideal fit for G.

Example



The min-weight matching for the following graph is 20.

The Marriage Problem

The marital problem is the one that follows. What conditions must be met in order to marry off the males such that each boy marries one of his girlfriends if we have a limited number of boys, each of whom has many girlfriends? (We presum that a female can only wed one boy).

Proof

The problem can be posed in graph-theoretical terms as follows:

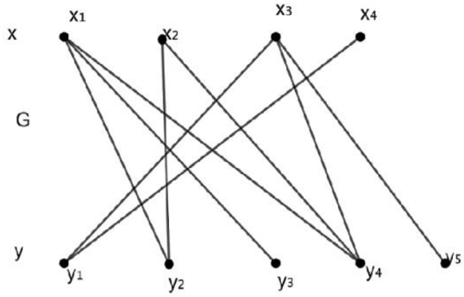
Construct a bipartite graph G with bipartition V (a) = XUY where $X = \{x_1, x_2 x_{3,} ....x_n\}$ represents the set of no boys and $Y = \{y_1, y_2 ....y_m\}$ represents their girlfriends, M in all.

An edges joins vertex x_i to y_i if y_1 is a girl friend of x_1. The marriage problem is then equation to finding condition for the existence of a matching in G which saturates every vertex of X.

For Example

If there are four boys x_1, x_2, x_3, x_4 and five girls $y_1, y_2, y_{3,}, y_4, y_5$ and relationship are given by

Boys	Girl friend
x_1	y_2 ,y_3 ,y_4
x_2	y_2 ,y_4
x_3	y_1 ,y_4 ,y_5
x_4	y_1



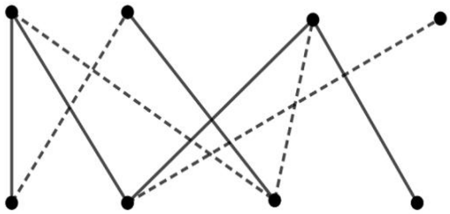
A bipartite graph for the girlfriend / boys friend example

A particular solution in the example is for x_3 to marry y_4, x_4 to marry y_1, x_1 to marry y_2 corresponding to the matching shown in figure.

A restatement of the general problem is let G be a bipartite graph with bipartition V (G) = X U Y.

Find the necessary and sufficient conditions for a matching in G to exist that saturates each vertices of X.

We shall provide the answer to the issue as it was first posed by Philip Hall in 1935.



In the graph G, the set of all the guys' girlfriends was indicated by N(S) if S is a subset of the boys.

Theorem: 1.22

(Hall's Marriage Theorem)

Let G be a bipartite graph with bipartition V = X U Y Then G condition a matching that saturates every vertex in X if and only if $|N(S)| \leq |S|$ for every subset S of X.

Proof

We refer to the set X set of boy's Y as the set of girls.

Let n + $|X|$ then for any K, 1<=K<=N, given any subsets S of K boys, to marry them all off to girlfriend they must have at least k girlfriend

i.e., $|N(S)| \geq k_1$

So $|N(S) \geq |S|$

If there is a solution to the marriage problem i.e., If there is a matching in G that saturates each vertex of X, then condition must be (*) satisfied.

Conversely Suppose $|N(S)| \geq |S|$ for every subset S of X i.e., to prove that every boy can be married off to one of his girlfriends.

We have perhaps lost one of their girlfriends, namely the one married to the first boy, therefore we may have lost "the girl left over," but condition (*) is still true for the subset of n-1 boys. Given any subset consisting of k boys from the remaining n-1,

We can marry each of these n-1 lads off to one of their girlfriends, which is the induction assumption.

As a result, we have married off all n of boys.

Case (1)

Suppose now that there is a set of k boys with k < n who collectively know exactly k girls then, since k < n, our induction hypothesis allows us to marry off these k boys, leaving n-k boys still number married but any collection of h boys from these n- k (with I ≤ h ≤ n- k) must have at least h girl friends from among the together with the above collection of k (h + k boys in all) would have less than h +k girlfriend, contrary to condition (*).

As a result, we may marry off each of the n - k remaining boys to a lady buddy using our induction hypothesis because condition (*) also applies to them.

Each of the n guys has so been successfully married off.

This demonstrates that, even if we assume the result is true for 1.....n-1, we can demonstrate that it is true for n, and therefore we may conclude that the result is true for all values of n.

Corollary

Let G be a k- regular bipartite graph with k > 0 then G has a perfect matching.

Proof

Let G have bipartition V = X U Y there are $|X|$ vertices in X.Each of these vertices has k edges incident with it.

Thus there are K X $|X|$ edges going from X to Y since each of the $|Y|$ vertices in Y to X by the incident with it.

Thus there are k* $|Y|$ edges going from Y to X by the bipartite nature of G each edge gone from X and Y end.So, G has K$|X|$ = K$|Y|$ edges.

Since K > 0 and K$|X|$ = K$|Y|$

$|X| = |Y|$

Now let S be a subset of X

Let f_1 denoted the set of edges incident with vertices in s then by the k-regularity of a $|E_1| = K|S|$. Let E_2 denote the set of edges incident with vertices with vertices in N(S). Since N(S) is the set of vertices which are joined by edges to s.

We have F_1 C E_2. Thus $|E_1|$ ☐ $|E_2|$

More over by the k- regularity of G we have $|F_2| \leq |E_2|$

Using 1, 2, 3, We get $K|N(S)| = |E_2| \leq |E_2| = K|S|$

So $K|N(S)| \leq K|S|$

Since K ☐ 0 this gives $|N(S)| \leq |S|$

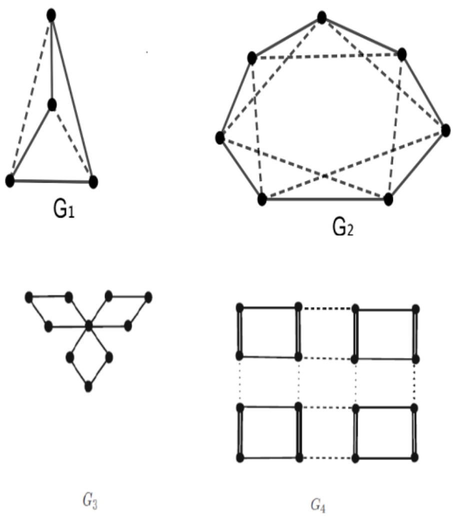
Since S was an arbitrary subset of X it follows from Hall's Marriage theorem that G contains a matching M also saturates every vertex in X. Since $|x| = |Y|$ the edges in the matching M also saturates every vertex in V. Thus M is a perfect matching in G.

Factorization

Definition: 1.23

A spanning subgraph of G is a factor of G. A graph's k-factor is a factor whose constituents are all k-regular graphs. Every component in a two-factor system is a cycle, but every component in a one-factor system is obviously k2. If there is an edge disjoined union of k-factorable in a graph G, then G is k-factorable.

Example: Graphs illustrating factorability



G_1 is 1-factorable

G_2 is 2-factorable

G_3 has neither a_1 factor nor is 2-factor

G_4 has 1 factor & 2- factor

Theorem: 1.23 (Tuttle's 1 factor theorem**)**

A graph G has 1-factor if and only if O (G-S) ≤ |S| for all S ⊆ V.

Proof

Only the adjacency of pairs of vertices is of importance to us when considering matching in a network. Without losing generality, we can suppose that G is a simple number.

If G has a 1-factor m, then each of the odd components of G-S must have at least one vertex which is to be matched only to a vertex of S under M

Hence for each odd component of G-S there exist an number of vertices in S should be at least as large as the number of components in G-S that is

O (G-S) ≤ |S| Conversely assume that condition (*) holds if G has no 1-factor we join pairs of non-adjacent vertices of G until we get a maximum sub graph G* of G until we get q□ having no 1-factor, Condition (*) holds clearly for G* since O(G* -S) ≤ O(G-S) by ((*) (*))

Taking S= Φ in (*)

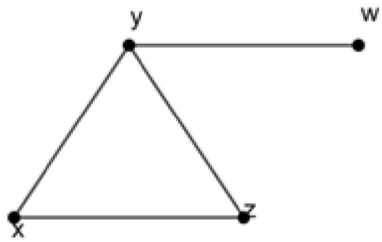
We see that O (G) = 0 and so n (G) (= n (G)) = n is even. For every pairs of non -adjacent vertices u and of G*, G* + u v has a 10 factorable

And any such 1-factorable must necessary contains the edges u v.

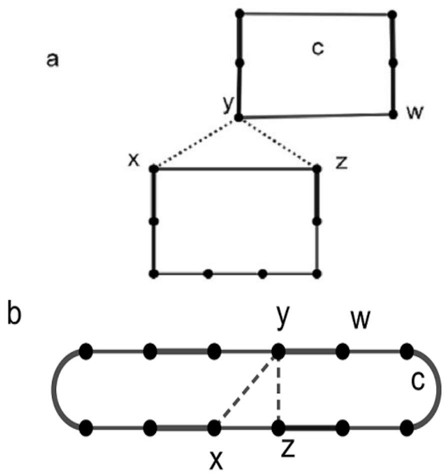
Let k be the set of vertex of G*of degree (n-1) K ≠ V

Since otherwise G* - K has perfect matching we claim that each component of G* - K is complete.

Suppose to the contrary that component G_1 of G* - K is not complete. Then in G_1 there are vertices x, y and z such that $xy \in E(G*)$, $yz \in E(G*)$ but x z does not belong to $E(G*)$ Moreover, since y $\in v(G_1)$, $d_{G^*}(y)$ □ n – 1 hence there exist a vertex W of G* with $yw \notin E(G*)$ necessary, w does not belong to k.



Super graph $G*$ for proof of theorem unbroken lines correspond to edge of $G*$ and broken lines correspond to edges not belong to $G*$.



(a) 1- factors M_1 and M_2 for (a)case 1 and (b) case 2 in proof of theorem ordinary lines correspond to edges of M_1 and bold lines correspond to edges of M_2.

By the choice of $G*, G*+xz$ and $G*+yw$ have 1-factors, say M_1 and M_2 respectively,

Necessarily $xy \in M_1$ and $yw \in M_1$.

Let H be the sub-graph of $G*+\{xz, yw\}$ induced by the edges in the symmetric difference $M_1 \Delta M_2$ of M_1 And M_2 since M_1 and M_2 Are 1-fector, each vertex of $G*$ is saturated by both M_1 and M_2 H is disjoint union of even cycle in which the edges at least alternative between M_1 and M_2.

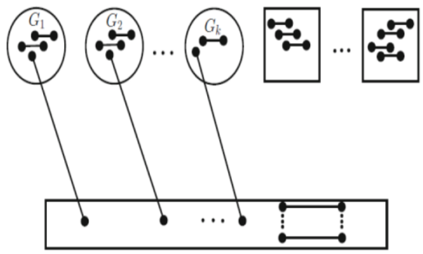
Case 1:

xy and yw belong to different components of H. If you belong to the cycle C, then the edges of M_1 in C together with the edges of M_2 not belonging to C from a 1-factor in G*Contradicting to the choice of G*

Case 2:

xz And yw belong to the same component C of H.

Since each component of H is a cycle C is a cycle by symmetry of x and y, we may suppose that the vertices x, y, w and z occur in that order on C from a 1- factor of $G*$ again contradiction the choice of $G*$



By condition (**), 0(G*-K) ≤ |K|

Hence, no vertices of each of the odd components of $G*$ - k is matched to vertex of k also the remaining vertices in each of the odd and even components of G*- k can be matched amongst themselves. The total number of vertices thus matched is even.

Since $|V(G*)|$ is even, the remaining vertices of k can be matched with almost these. This gives a 1-factor of G* but the choice has no- 1- factor this contradiction proves that G has 1-factors.

Theroem: 1.24

Every 3-regular bridgeless graph contains a 1-factor.

Proof

Let G be a 3-regular bridgeless graph and let S be a subset of $V(G)$ cardinality k > 1.

We show that the number $K_0(G-S)$ of odd component of $G-S$ we may assume that $G-S$ has 1≤ 1 odd components $G_1, G_2 G_3 ...... G_l$.

Let $x_i (1 \le i \le 1)$ denotes the set of edge joining the vertices of S and the vertices of G_i.

Since every vertex of each graph G_i has degree 2 in G and the sum of the degree of the vertices in the graph G_i is even $|x_i|$ is odd, because G is bridgeless, $|x_i| \neq 1$ for each i$(1 \leq i \leq l)$ so $|x_i| \geq 3$

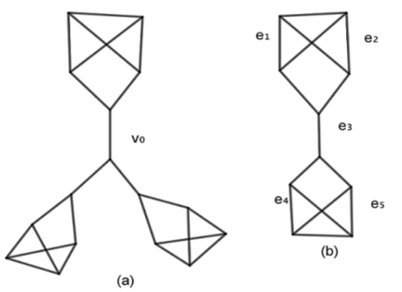
Therefore, there are at least 3 l edges joining the vertices of S and the vertices of G-S.

However, since $|s| = K$ and every vertex of S has degree 3 in G. At most 3k edges join the vertex of S and the vertices of $G - S$.

Therefore $3\,k_0(G-S) = 3l \leq 3k = 3|S|$ and so $k_0(G-S) \leq |S|$ by theorem, a graph G contains a 1- factor if and only if $k_0(G-S) \leq |S|$ for every proper subset S of G has a 1-factor. Hence the theorem is proved.

Example

A 3 –regular graph with cut edges may not have 1-factor having cut edges.



(a) By augural again a cubic graph with a 1-factor may have cut edges see fig (b).

In fig (a) if, $S = \{V\}$ $O(G-S)$=3 > $1 = |S|$, and so G has no 1-factor.

In the fig (b) $\{e_{1,} e_{2,} e_{3,} e_{4,} e_5\}$ is a 1- factor, and e_3 is cut edge of G.

If G has no 1- factor then by theorem, A graph G has 1 factor if and only if $O(G-S) \leq |S|$ there exist $S \subseteq V(G)$ with $O(G-S) > |S|$ such that S is called an anti factor set of G.

Let G has be a graph of even order n and let S be an anti-factor set of G.

Let $O(G-S) = K$ and $G_1, G_{2,......} G_K$ be the odd component of $G - S$ since n is even.

$|s|$ and k have the same parity.

Choose a vertex $u_i \in V(G_I), 1 \leq i \leq k$

Then $|\, S \cup \{u_1, u_2, \cdot.....u_k\}|$ is even, i.e., $k + |S \equiv 0(\text{mod } 2)|$

This means that $k \equiv |S(\text{mod } 2)|$

Thus $O(G-S) \equiv |S|(\text{mod } 2)$

Thus we make the following observation.

Observation

If S is an anti –factor set of graph G of even order then $O(G-S) \geq |S| + 2$

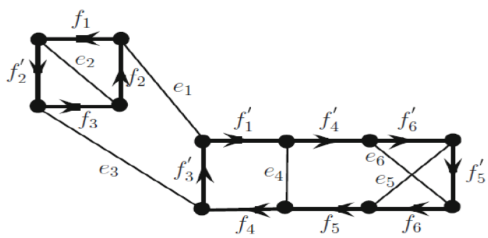
Corollary

A basic 2-edge linked cubic graph G's edge set may be divided into three pathways of equal length.

Proof

Each three-regular graph has a single component and no cut edges. G is the union of any two 1-factor or 2-factor orientations of the edges of the aforementioned 2-factor cycle.

So, that each cycle becomes a directed cycle. Then e is any edge of the 1-factor. $f_{1,} f_{2,}$,are the two arcs of G having their tails at the end vertices of e, then $\{e, f_1, f_2,\}$ forms a typical 3- path of the edges partition of G.



Corollary

A $(p-1)$ regular simple graph on 2p vertices has a 1-factor

Proof

Proof is by contradiction.

Let G be a $(p-1)$ regular simple graph on 2p vertices having no 1-factor.

Then G has an anti-factor set S.

By observation, $O(G-S) \geq |S|+2$

Hence

$|S|+(|S|+2\ ) \leq 2p$,

Therefore $|S| \leq p-1$.

Let $|S| = p-r$, then $r \neq 1$, since if r=1, $|S| = p-1$, and

Therefore $O(G-S) = P+1$

Hence each odd component of G-S is a singleton

Hence, each such vertex must be adjust to all the $p-1$ vertices of S but this means that every vertex of S is of degree at least $p+1$, Contradiction to the fact,

Hence $|S| = p-r, 2 \leq r \leq p-1$

If 'G' is any component of $G-S$ and $V \in V(G^1)$ then V can be adjacent to at most $|S|$ vertices of S. there four as G is $(p-1)$ regular must be adjacent to at least $(p-1)-(p-r) = r-1$ vertices of G^1.

Thus $|V(G^1)| \geq r$

Counting the vertices of S,

We get $(|S|+2\ )+|S| \leq 2p$

OR

$(p-r+2\ )r+(p-r) \leq 2p$

This gives $(r-1)(r-p) \geq 0$ violating the contradiction on r.

Claw Free

If a graph doesn't have an induced subgraph that is isomorphic to If every vertex has precisely theorem incident edges, a graph is said to be cubic. Every cubic graph with no cut edges has perfect matching, according to a well-known Peterson classical theorem.

THOMAS: 1.25

If a linked graph is claw-free and has a 1-factor, let it be of even order n.

Proof

Contains a minimum anti-factor set of if there is no 1-factor. Every odd component of must have a border around it.

If $V \in S$ and V_x, V_y, V_z are edges of G with x, y, z belonging to distinct odd component of G then $k_{1,3}$ is induced in G by hypotheses, this is not possible.

Since $O(G-S) > |S|$ there must certainly exist a vertex V of S, and edges uv and vw of G with u and w in distinct odd components of $G-S$

Suppose G_u and G_w are the odd component of $G-S_1$ where $S_1 = S-\{V\}$.

Further

$O(G-S_1) = O(G-S)-1 > |S|-1 = |S_1|$

Hence s_1 is an anti-factor set of G with $|S_1| = |S|-1$, a contradiction to the choice of S.

Thus G must have 1-factor.

One of the few areas of mathematics with a known birth date is graph theory. The publication of Euler's solution to the Königsberg bridge puzzle in 1736 is regarded as the birth of modern graph theory. A graph is any mathematical entity that consists of points and the relationships between them. A graph G is made up of a set E (G) of 2-element subsets of V(G) called edges and a nonempty set V (G) of objects called vertices. The vertex set of G is designated as V (G), and its edge set as E (G). The size and order of the graph G are determined by how many edges there are. Graphs of order p and size q are referred to as graph.

Graphs may be used to illustrate a wide range of real-world scenarios. The study of proving methods in discrete mathematics may be enjoyed via the lens of graph theory, whose findings have wide-ranging applications in the social, natural, and computer sciences. Edge weights may be used to model the road network and be used to calculate trip time or distance.

The identification of a graph's elements, such as its vertices or edges, by some practical technique of addressing them, is a common step in issues of the type mentioned above. This area of graph theory issues is referred to as a set of graph labeling issues.

Though sometimes driven by practical reasons, graph labeling—in which the vertices and edges are given components of a specific set or subsets of a set subject to certain conditions—is also intriguing in and of itself. The first introduction of graph labeling techniques occurred in the middle of the 1960s. The majority of graph labeling techniques have their roots in one Rosa introduction from 1967. Solomon W. Golomb is responsible for the term "graceful labeling." Alex Rosa first gave this category of labeling the word "labeling."

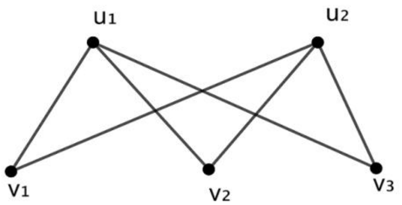
Around the subject, there is a vast body of literature. More than 650 publications on various graph labeling techniques have been published in the past 30 years or more, indicating distinct types of graphs that may accept a certain form of labeling. For a variety of applications, the family of mathematical models known as labeled graphs is becoming more and more valuable.

Basic Definition

1. Complete Bipartite Graph

Let (x, y) be a bipartite graph. If each vertex of x is adjacent to each vertex of y then the graph is said to be a complete bipartite graph. If the vertex subset X & Y have m & n vertices then the complete biograph is denoted by $K_{m,n}$ or $K_{n,m}$.

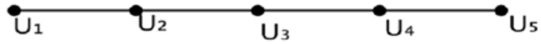
Example: $K_{2,3}$



2. Path Graph

A graph is called a path if the degree $d(v)$ of every vertex v, is ≤ 2 & there are no more than 2 end vertices. An end vertex or leaf is vertex of degree 1.

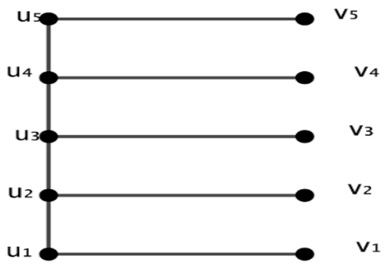
Example: P_5



3. Comb Graph

The graph obtained by joining a pendant edge at each vertex of a path P_n is called a comb & is denoted by $p_n\ K_1$ or $P_n^{\ +}$

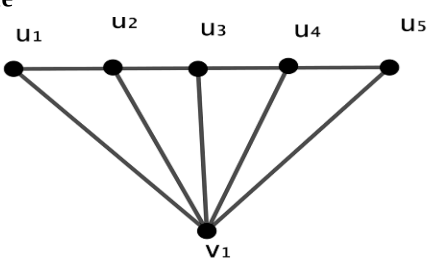
Example



4. Fan Graph

A fan graph $F_{m,n}$ is defined as the graph join $K_m + P_n$, where K_m is the empty graph on m nodes and P_n is the path graph on n nodes. The case $m = 1$ corresponds to the usual fan graphs, while $m = 2$ corresponds to the double fan.

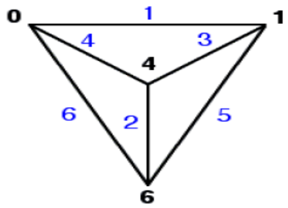
Example

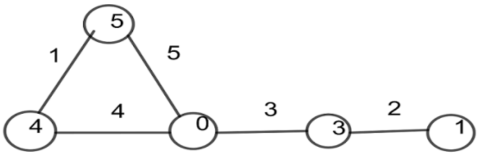


5. Graceful Labeling

A graceful labeling is a labeling of the vertices of a graph with distinct integers from the set $\{0.1.2,3........, q\}$ where q represents the number of edges such that if $f(v)$ denotes the label even to vertex v, when each edge uv is given the value $|f(u)-f(v)|$ the edges are labeled 1, 2……q.

Example





6. Even Graceful Labeling

A function f is called an even graceful labeling of a graph G with q edges. It f is an injection from the vertices of G to the set $\{0,2,...2q\}$ such that when each uv is assigned the label $|f(u)-f(v)|$ the resulting edge labels are distinct even numbers i.e.) ranges from 2 to 2q.

A graph which admits even graceful numbering is said to be even graceful graph. An even graceful labeling f is called an even alpha-valuating of G.

7. Odd - Even Graceful Labeling

The odd - even graceful labeling of a graph G with q edges means that there is an injection $f:V(G)\rightarrow\{1,3,5,...2q+1\}$ such that when each edges uv is assigned the label $|f(u)-f(v)|$ the resulting edge labels are $\{0,2,...2q\}$. A graph which admits an Odd-even graceful labeling is called an Odd-even Graceful Graph.

Application

Numerous areas rely heavily on the topic of graph theory. Graph labeling, one of the key topics of graph theory, is utilized in numerous fields, including data base administration, radar, astronomy, x-ray crystallography, circuit design, and communication network addressing. This study, while providing a general overview of labeling of graphs in various disciplines, mostly concentrates on communication networks. There are two different types of communication networks: wired communication and wireless communication. Wireless networks are being built daily to make it easier for any two systems to communicate, which leads to more effective communication. The effect of labeling in enhancing the usefulness of this channel assignment procedure in communication networks was also investigated in this research. Numerous articles using graph labeling have been discovered and its application to communication networks has been noted. The use of graph labeling to network security, network addressing, channel assignment, and social networks is discussed in this study. Here is a summary and some fresh suggestions.

A graph is defined as a pair G = (V, E), where V is the set of all vertices and E is the set of all edges. The elements of E are subsets of V that have precisely two members. If each edge of a graph G has the value f (UV) = f (u)*f (v), where * is a binary operation, then the graph is said to be labeled. * can be found in literary works to represent addition, multiplication, absolute difference, modulo subtraction, or symmetric difference [20]. In many areas of computer science, including data structures, graph algorithms, parallel and distributed computing, and communication networks, network representations are crucial. Global representations characterize traditional networks in most cases. That is, one must access a global data structure that represents the whole network in order to obtain relevant information. From social and communication networks to the Web, massive graphs are present everywhere. These data sets' geometric depiction of the enforced graph structure is a potent tool for interpreting and visualizing the data.

One of the most used approaches for labeling graphs is the odd graceful labeling [18]. The labeling of graphs is thought to be primarily a theoretical topic in the fields of discrete mathematics and graph theory, but it has a wide range of applications, some of which are listed below.

The coding hypothesis In order to construct some significant classes of effective non-periodic codes for pulse radar and missile guidance, the entire graph must be labeled in a way that ensures that each edge has a unique label. The time locations at which pulses are sent are then determined by the node labels.

Radiation crystallography One of the most effective methods for identifying the structural characteristics of crystalline solids is X-ray diffraction, in which an X-ray beam impacts the crystal and diffracts into a variety of distinct orientations. Sometimes, the same diffraction data is present in many structures. This issue is mathematically identical to figuring out all of the labeling for the relevant graphs that result in a predetermined set of edge labels.

Addressing the communication network: A communication network is made up of nodes, each of which has computer power and can send and receive messages across wired or wireless communication links. A few examples of the fundamental network topologies are completely linked, mesh, star, ring, tree, and bus. A single network may be made up of a number of linked subnets with various topologies. In this work, these difficulties are briefly examined.

Local Area Networks (LAN), which include networks within a single building, and Wide Area Networks (WAN), which include networks between buildings, are further categories for networks. Giving each user terminal a "node label" can be beneficial, provided that all connected "edges" (communication connections) get unique labels. In this method, the numbers of any two communicating terminals immediately identify the link label of the connecting path (by simple subtraction), and vice versa, the label of the connecting path specifically identifies the pair of user terminals that it links.

Role of Graph Labeling

Numerous applications are mentioned here. Depending on the problem scenario, a certain type of graph is employed for each application type to depict the issue. The challenge is solved by applying an appropriate labeling to that graph. Numerous concerns are discussed after quickly and effectively establishing contact.

Fast Communication in Sensor Networks Using Radio Labeling

Each station is given a channel (a positive integer) so that interference may be avoided given a group of transmitters. The interference grows stronger the closer the stations are together, necessitating a bigger disparity in channel assignment. Here, every vertex stands in for a transmitter, and any two vertices joined by an edge signify nearby transmitters. Let G = (V (G), E(G)) be a connected graph and let d(u, v) be the distance between any two vertices in G. This type of labeling is known as radio labeling. The diameter of G, given by diam, is the greatest distance possible between any two vertices.

A radio labeling (or multilevel distance labeling) for G is an N union {0} such that for any vertices u and v, |f (u) −f (v)|◊injective function f: V (G) >=diam (G) − d (u, v)+1.

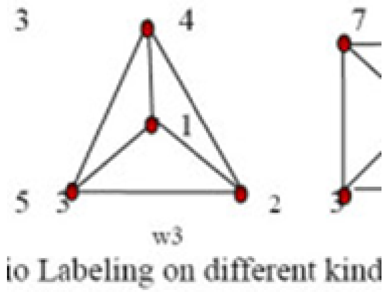


Fig1: Radio Labeling on different kind of Graphs
Click here to View Figure

Each station is given a channel (a positive integer) so that interference may be avoided given a group of transmitters. The interference grows stronger the closer the stations are together, necessitating a bigger disparity in channel assignment. Here, every vertex stands in for a transmitter, and any two vertices joined by an edge signify nearby transmitters. Let G = (V (G), E(G)) be a connected graph and let d(u, v) be the distance between any two vertices in G. This type of labeling is known as radio labeling. The diameter of G, given by diam, is the greatest distance possible between any two vertices.

Designing Fault Tolerant Systems with Facility Graphs

Each station is given a channel (a positive integer) so that interference may be avoided given a group of transmitters. The interference grows stronger the closer the stations are together, necessitating a bigger disparity in channel assignment. Here, every vertex stands in for a transmitter, and any two vertices joined by an edge signify nearby transmitters. Let G = (V (G), E(G)) be a connected graph and let d(u, v) be the distance between any two vertices in G. This type of labeling is known as radio labeling. The diameter of G, given by diam, is the greatest distance possible between any two vertices. Numbers in parenthesis are used to denote the kind of facility; this process is known as graph vertex labeling. The graph shows the different kinds of facilities that other facilities can access. The nodes x2 and x4 are accessed through node x1. In a similar way, the node x5 with facility type t1 may access nodes x3, x2, and x4's respective facility types t3, t1, and t2. When a node in this facility graph fails, there is no need to worry about the communication link since the facility graph will discover a new way and continue the communication process as usual.

Automatic Channel Allocation for Small Wireless Local Area Network

Safe transmissions are required in many domains, including cellular phone, Wi-Fi, security systems, and others, to discover an effective solution. Being on the phone and having another caller pick up is annoying. Interferences brought on by unrestricted simultaneous broadcasts generate this annoyance. Communication damage can result from interference or resonance between two near enough channels. With the right channel assignment, the interference may be avoided.

The channel assignment problem is the issue of allocating a channel, a nonnegative number, to each TV or radio transmitter spread out over different locations without interfering with communication. In a graph representation of this issue, the transmitters are represented by a graph's vertices; two vertices are considered to be close if they are two vertices away or extremely close if they are contiguous in the graph.

Close transmitters must receive distinct channels in a private conversation, and extremely close emitters must receive channels that are at least two channels apart.

The network wireless LAN is modelled as an interface graph to handle this issue, and the graph labeling approach is used to resolve it.

The access points (vertices) in the interference graph interfere with some other access points in the same area. The access points act as nodes to create a network known as an interference graph. If the nodes interact with one another while utilizing the same channel, an undirected edge is linking them. The channel allocation problem has now been transformed into a vertex labeling problem, which is a graph labeling problem.

The channels on the access points are represented by the set of colors C. The margins of these channels should ideally not overlap. For labeling purposes, the DSATUR (Degree of Saturation) labeling algorithm is utilized. The technique uses a heuristic search, which identifies vertices with the greatest number of neighbors with various colors. This A vertex coloring function f: v (G) subset is picked for labeling if it only includes one vertex. If the subset has more than one vertex, the labeling is done in the order of the neighbors with the fewest unlabeled edges. A deterministic selection function is used to choose the vertex in place of the final selection if there are several candidate vertex options available. The protocol operation involves listening for messages produced by the access points in order to identify the neighbors. When a message is broadcast again by the access points, the protocol action is complete. After completing this, the labeling algorithm is used to generate the interference graph.

The channels and the graph are analogous in that the labeling process should be executed at regular intervals, just as the channels listen to the messages at regular intervals.

Avoiding Stealth Worms by Using Vertex Covering Algorithm

The vertex cover algorithm looks for a vertex cover with a maximum size of k given a basic graph G with n vertices labeled 1, 2,... n as input. If the vertex cover achieved at each stage has a size of at most k, halt.) is employed to model the spread of stealth worms on big computer networks and create the best defenses against viral assaults in real time. Finding the worm propagation is crucial since it allows you to stop it in its tracks.

Finding a minimal vertex cover in the graph, where the edges are the links between the routing servers and the vertices are the routing servers, is the fundamental concept used here. Then worm propagation's ideal solution is discovered.

Analyzing Communication Efficiency in Sensor Networks with Voronoi Graph

The applications for sensor networks are numerous. tracking of moving objects, gathering environmental data, defense uses, medical applications, etc.

To examine the effectiveness of communication, a graph model of the sensor network is used. In this case, the sensor network is modeled using a voronoi graph. Voronoi graphs are built in the shape of polygons on a plane, with the nodes acting as sensors and the edges of the polygons serving as the sensing range of each sensor.

The detecting area of these sensors may be thought of as the polygon. One of these sensors will serve as the reporting function's cluster head. If the detecting ranges of two sensors on a voronoi graph have a shared border, they are said to be neighbors.

The preceding sensor should appropriately notify the adjacent sensor when an item enters the sensing range of another sensor after crossing the border of one sensor, or the sensing range of one sensor.

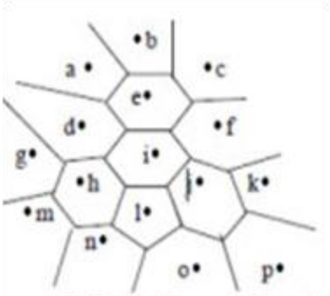


FIG 3. Vornoi Graph

FIG 3: Voronoi Graph:
Click here to View Figure

The detecting area of these sensors may be thought of as the polygon. One of these sensors will serve as the reporting function's cluster head. If the detecting ranges of two sensors on a voronoi graph have a shared border, they are said to be neighbors.

The preceding sensor should appropriately notify the adjacent sensor when an item enters the sensing range of another sensor after crossing the border of one sensor, or the sensing range of one sensor.

Reducing the Complexity of Algorithms in Compression Networks

The detecting area of these sensors may be thought of as the polygon. One of these sensors will serve as the reporting function's cluster head. If the detecting ranges of two sensors on a voronoi graph have a shared border, they are said to be neighbors.

The preceding sensor should appropriately notify the adjacent sensor when an item enters the sensing range of another sensor after crossing the border of one sensor, or the sensing range of one sensor. To distinguish it from algorithmic graph compression, when a graph is compressed to lessen the time or space complexity of a graph algorithm, this method is known as graph compression, or more precisely, semantic graph compression.

To lessen the time or space complexity of a graph method, a graph is compressed in Compression Networks. The creation of a communication network has various benefits. One benefit is that each connection is tagged with the difference between the two communication centers, so if a link breaks, a straightforward algorithm can identify which two centers are no longer connected.

Another benefit is that this network would have all the characteristics of a graceful graph, including the ability to undergo cyclic decompositions.

Graph Labeling in Communication Relevant to Adhoc Networks

Mobile Adhoc Networks problems can also be addressed with graph labeling. Issues including connection, scalability, routing, network modeling, and simulation must be taken into account in ad hoc networks. The model may be used to examine these problems since a network can be represented as a graph. Matrix representations of graphs can be created algebraically. Additionally, algorithms may be used to automate networks. It is necessary to mimic difficulties such node density, mobility among nodes, connection construction between nodes, and packet routing. Random graph theory is employed to replicate these ideas. The analysis of congestion in MANETs, where these networks are modeled using graph theoretical concepts, is also possible using a variety of techniques.

Effective Communication in Social Networks by Using Graphs

The communication networks centered on individuals are known as social networks. These include both conventional and contemporary social networks.

Traditional Social Networks

Sociologists and other humanities academics have been examining the organization of social groupings for a very long time, long before the Internet began to affect the lives of many people. The majority of the time, relatively small groups were taken into consideration because it was frequently impractical to analyze big groups.

Traditional networks are employed to determine the significance of certain individuals or groups. A person with many ties to others could be viewed as being relatively significant. The drawback of these networks is that a person in the center appears to have greater influence than someone at the perimeter. Modern social networks are offered as online communities in order to address the shortcomings of conventional social networks.

Online Communities

The Internet has made it possible for users to communicate with one another via user-to-user messaging platforms. E-mail is the most well-known of these systems and has existed since the inception of the Internet. Another well-known example is network news, which allows users to post comments on online message boards and then allows others to respond, resulting in debate threads of all shapes and sizes.

Instant messaging systems have gained popularity more lately, enabling users to communicate immediately and interactively with one another, maybe augmented with knowledge of their respective present states. Almost 2 million emails are sent every second by the more than 1 billion consumers that use email today.

Sociograms made a significant contribution to social network analysis. A sociogram may be thought of as a visual depiction of a network, with persons represented by dots (known as vertices) and their connections by lines.

It's fascinating to observe how these communication tools affect the users. People who have never physically met each other utilize online communities to share ideas, opinions, feelings, and other information. This online community is referred to as a little world. Every two persons may contact each other through a chain of only a few messages, which defines a tiny world. It is the occurrence of messages moving via an email network.

Users are connected by virtue of knowing one another, and the resulting network has small world characteristics, essentially linking each user to every other user through relatively short chains of such relationships. The core of network science consists in describing and characterizing these networks as well as others.

Secure Communication in Graphs

Only when the messages are verified can secure communication be accomplished in the face of a malevolent adversary on an open and dynamic network. Channels for authentication are utilized for this purpose. Such channels can be established in a number of ways. Here, for instance, shared secret keys or public keys are utilized with dedicated network communication lines.

By Using Certificates

A trust graph is a graph with the network's processors as its vertices and the authentication channels as its edges. The messages may be verified over the associated channel if the sender and recipient are linked by that edge in the graph. Otherwise, the trust graph's intermediate processors are employed for authentication pathways.

Consider the issue of secure communication in a network with malicious flaws where the adversary may not be aware of the trust graph, which has processors as nodes and certified public keys as edges. In various models, this scenario happens. For instance, in models of survival where certifying authority might be compromised, or in networks that are being built decentralized.

If the trust graph is sufficiently linked, a protocol is supplied that enables safe communication in this situation.

By Using Key Graphs

A crucial networking challenge will be how to secure group communications, i.e., how to guarantee the secrecy, authenticity, and integrity of messages sent between group members. Secure groups are defined using key graphs. Three solutions are shown here for securely disseminating rekey messages after a join/leave and provide protocols for entering and exiting a secure group for a certain class of key graphs.

A collection of users, a set of keys owned by the users, and a user-key relationship are the three techniques. It is scaleable to big groups with frequent joins and leavers by employing one of the three rekeying procedures. In instance, the logarithm of group size results in a linear rise in the average recorded processing time for each join and exit.

Identification of Routing Algorithm With Short Label Names

Nodes in graphs are labeled using informative labeling schemes such that questions about any two nodes, such as whether they are neighboring or not, may be answered by looking only at the labels of the associated nodes.

Such approaches aim to reduce label size, or the maximum number of bits that may be contained in a label.

For the adjacency and ancestry issues on trees, several probabilistic one-sided error schemes are built using probabilistic labeling techniques. While others are paired with suitable lower limits demonstrating that, with the resultant assurances of success, one cannot expect to perform much better in terms of label size, several of the schemes greatly improve the bound on the label size of the equivalent deterministic schemes.

Automatic Routing with Labeling

If a certain type of graph topology can be used to represent any conventional network, labeling applied to the network may automatically determine the path and any other information.

Any type of fixed graph structure is acceptable in this case, including cycles, wheels, fan graphs, and buddy graphs. Now that a magic constant is made visible to the network, magic labeling may be used. Now, the router uses the magic constant, its own label, and labels associated with channels to automatically identify the next node to be reached. Since a magic number must be produced by all of these.

Security with Reducing the Packet Size Using Labeling Schema

Any type of network must have an agreement between the source and the destination for this application. They should both wear the appropriate labels. Naturally, it might not apply to all packs, but the bulk of suits. The packet's information has to match the agreement's labeling style.

A packet's information is applied with labeling, and if it satisfies the magic labeling requirements for a certain magic constant (k), the packet will have 1, k labels, one for each vertex, allowing for the transmission of half of the data. 1 denotes labeling for a packet; k is a magic constant and represents packet data. Because the packet does not include all the bits, security is also provided. The original packet information can only be retrieved by the destination that has the information of agreement. With this application, the packet transformation is quick due to the smaller size and greater security of the missing data.

Conclusion

Exploring the function of graph labeling in the communication sector is the primary goal. As mentioned above, Graph Labeling is a strong tool that simplifies tasks in a variety of networking-related domains. A summary is provided specifically to illustrate the concept of Graph Labeling. Researchers may learn more about graph labeling and its uses in the communication industry, as well as get insight into potential directions for their own study.

References

1. Bermond J.C, **Graceful graphs, radio antennae and French windmills**, Graph theory and Combinatorics, Pitman, London, (1979), 13 – 37.
2. Balaji V, Ramesh D.S.T and Subramanian A, **Skolem Mean Labeling,** Bulletin of Pure and Applied Sciences, vol. 26E No. 2, 2007, 245 – 248.
3. Balaji V, Ramesh D.S.T and Subramanian A, **Some Results on Skolem Mean Graphs,** Bulletin of Pure and Applied Sciences, vol. 27E No. 1, 2008, 67 – 74.
4. Balaji V, Ramesh D.S.T and Subramanian A, **Relaxed Skolem Mean Labeling,** Advances and Applications in Discrete Mathematics, vol. 5(1), January 2010,1–22.
5. Balaji V, Ramesh D.S.T and Subramanian A, **Some Results On Relaxed Skolem Mean Graphs,** Bulletin of Kerala Mathematics Association, vol. 5(2) December 2009, 33 – 44.
6. Balaji V, **Solution of a Conjecture on Skolem Mean Graph of stars** $K_{1,\ell} \cup K_{1,m} \cup K_{1,n}$, International Journal of Mathematical Combinatorics, vol.4, 2011, 115 – 117.
7. Balaji V, Ramesh D.S.T and Maheswari V, **Solution of a Conjecture on Skolem Mean Graph of Stars** $K_{1,\ell} \cup K_{1,\ell} \cup K_{1,m} \cup K_{1,n}$, International Journal of Scientific & Engineering Research, 3(11) 2012, 125 – 128.
8. Balaji V, Ramesh D.S.T and Maheswari V, **Solution of a Conjecture on Skolem Mean Graph of Stars** $K_{1,1} \cup K_{1,1} \cup K_{1,m} \cup K_{1,n}$, Sacred Heart Journal of Science & Humanities, Volume 3, July 2013.
9. Bloom G.S, Golomb S.W, **Application of numbered undirected graphs,** Proceeding IEEE 65 (1977) 562–570.
10. Devaraj J, **A study on different classes of graphs and their labeling,** Ph.D. thesis, University of Kerala, India 2002.
11. Gallian J.A, **A dynamic survey of graph labeling**, The Electronic Journal of Combinatorics, 6 (2010), # DS6.
12. Graham R. Land Sloane N.J.A, **On additive bases and harmonious Graphs,** SIAM J. Alg. Discrete meth, 1 (1980) 382 – 404.
13. Harary F, **Graph Theory**, Addison – Wesley, Reading, 1969.
14. Liu Bolian and Zhang Xiankun, **On harmonious labeling of graphs**, Ars Combinatoria 36 (1993), 315 – 326.
15. Maheswari V, Ramesh D.S.T and Balaji V, **Relaxed Mean Labeling,** International Journal of Mathematics Research Volume 4, Number 3 (2012), Page No. 217 – 224.
16. Maheswari V, Ramesh D.S.T and Balaji V, **On Relaxed Mean Labeling,** Proceeding of National Seminar on Recent Trends in Mathematics (ISBN NO: 978-93-80686-35-6), Aringnar Anna College, Krishnagiri, March 1 (2015), 119 –130.
17. Maheswari V, Ramesh D.S.T and Balaji V, **Some Results on Relaxed Mean Labeling,**

International Journal of Mathematical Combinatorics (**Accepted**).

18. Maheswari V, Ramesh D.S.T, Silviya Francis and Balaji V, **On Mean Labeling,** Bulletin of Kerala Mathematics Association Volume 12,No.1 June 2017,54–64.

19. Maheswari V, Ramesh D.S.T, Silviya Francis and Balaji V, **Non Existence of Relaxed Mean Labeling for Subdivision of Star Graphs,** Presented a paper in **National Conference on Recent Developments in Coloring, Domination and Distinguishing Codes in Graphs**.

20. Maheswari V, Ramesh D.S.T and Balaji V, **On Skolem Mean Labeling**, Bulletin of Kerala Mathematics Association, Vol. 10, No.1, 2013, 89 – 94.

21. A Textbook of Graph Theory - PDF Free Download. https://epdf.tips/a-textbook-of-graph-theory.html

22. Graph Theory II 1 Matchings - MIT. http://web.mit.edu/neboat/Public/6.042/graphtheory2.pdf

23. Erasmus/CEEPUS Lecture. https://people.inf.elte.hu/szabolaszlo/sm.pdf

24. Answered: A graph G= (V, E) is planar, if it can… | bartleby. https://www.bartleby.com/questions-and-answers/a-graph-g-v-e-is-planar-if-it-can-be-drawn-in-the-plane-so-that-no-edge-pairs-cross.-how-many-colors/6ec3346f-1c7d-44b0-a093-7aae0dc52917