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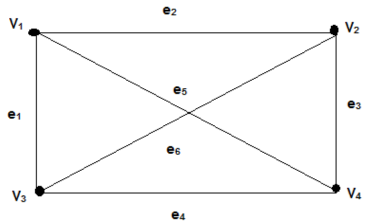
Definition

A graph G consists of a pair *(V, E)*, where *V* is a non-empty finite set whose elements are called vertices(points) and *E* is a set of unordered pair of distinct elements of *E* are called edges (line or link) of thegraph G.

Example



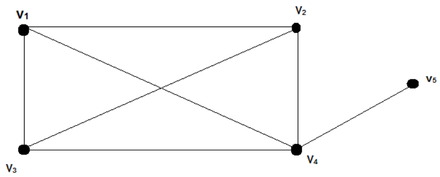
(A graph with 4 vertices and 6 edges)
V_1, V_2, V_3, V_4 are vertices
$e_1, e_2, e_3, e_4, e_5, e_6$ are edges.

Definition

The degree of the vertex *v* in a graph *G* is the number of the edges incident with *V*. The degree of the vertex *v* is denoted by deg*(v)* or *d(v)*.

The minimum degree and the maximum degree of a graph of G are usually denoted by special symbol & (G) and Δ (G) respectively.

Example

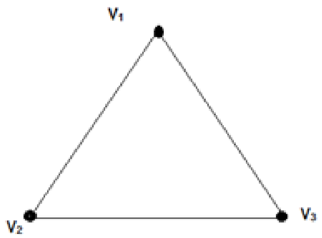


deg (V_1) = 3
deg (V_4) = 4

Definition

A graph that has neither self - loops nor parallel edges is called a simple graph.

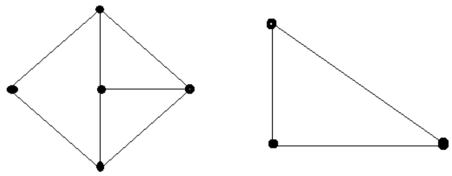
Example



Definition

A graph G_1 is said to be a subgraph of graph G if all vertices and allthe edges of G_1 are in G and each edge of G_1 has the same end vertices in G_1 as in G.

Example

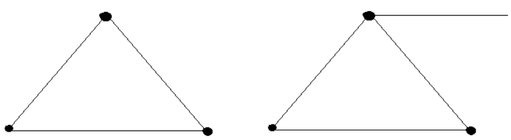


G **G_1**

Definition

A graph G is said to be connected. If there is at least one path between every pair of vertices in G. Otherwise G is disconnected.

Example



Connected **disconnected**

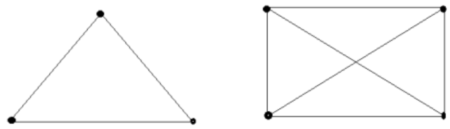
Definition

A disconnected graph consists of two or more connected graphs. Each of these connected sub graph is called component.

Definition

A simple graph in which there exists an edge between every pair of vertices is called complete graph. The complete graph with n- vertices is denoted by K_n.

Example

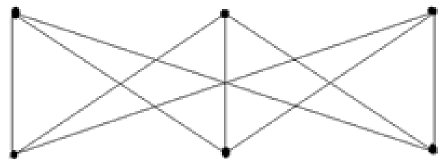


K_3 **k_4**

Definition

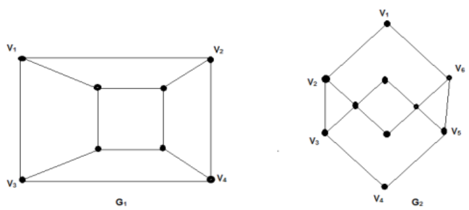
A graph G is called bipartite if its vertex set V can be decomposed into disjoint subsets V_1 and V_2 such that every edge in G join a vertex in V_1 with a vertex in V_2.

Example



Definition

Two graphs G and G' are said to be the isomorphic to each other if there is a one to one correspondence between their vertices and between their edges such that incidence relationship is preserved.



Definition

A walk is defined as a finite alternating sequence of vertices and edges beginning and ending with vertices such that each edge is incident with the vertices preceding and following it.

Definition

A walk is closed if it has positive length and its origin and terminal are the same.

Definition

A walk that is not closed is called on open walk.

Definition

An open walk in which no vertex appears more than one is called path.

Definition

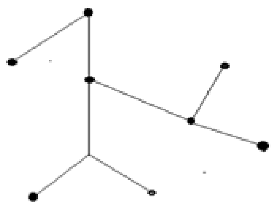
The number of edges in a path is called length of a path.

Example:

Definition

A tree is a connected graph without any cycles.

Example

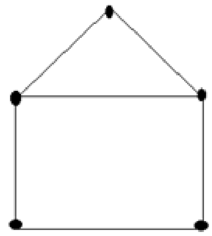


Result

A tree with n- vertices has (n -1) edges.

Definition

A closed walk in which no vertex appears more than once is called a cycle.



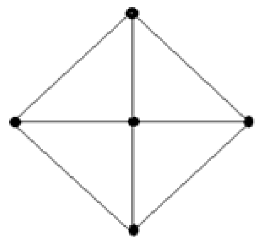
Result

A graph G with n-vertices is called tree if G is minimally connected graph.

Definition: 1

A complete graph is sometimes also referred to as universal graph or clique.

Example



Definition

The size of the largest clique in G is denoted by $W(G)$. In the above fig $W(G) = 4$

Definition

In connected graph G the distance $d\ (v_i, v_j)$ of two of its vertices v_i and v_j is the length of the shortest path between them.

Result

The graph G with n – vertices is called tree if G is minimally connected graph.

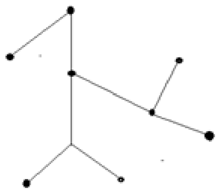
Definition

The eccentricity $e(v)$ of the vertex v. In a graph G is the distance from v to the vertex farthest from v in G.

Definition

The diameter of the graph G is defined as minimum eccentricity among all vertices of the graph.

Example



The diameter of the graph *G* is 4

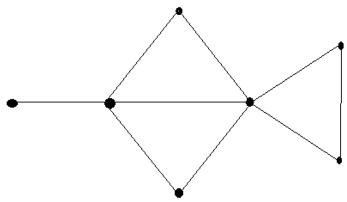
Definition

A grapy is said to be regular if all its vertices have the same degree otherwise it is called non – regular.

Definition

A set of vertices in graph is said to be an independent set if no two vertices in the set are adjacent.

Example



$\{v_1, v_2, v_3, v_4\}$ is independent set

Definition

A maximal independent set is an independent set to which no other vertex can be added without destroying its independence property.

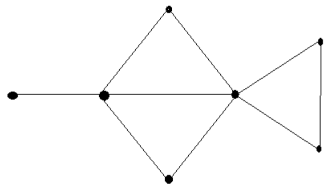
The set $\{v_1, v_2, v_3, v_4\}$ is maximum independent set in the above figure.

The set $\{v_2, v_6\}$ is another maximal independent set.

Definition

The number of vertices in the maximal independent set of a graph of G is called the independent number. It is denoted by β (G).

Example



Maximal independent set = $\{v_2, v_5\}$
Therefore $\beta(G) = 2$

Definition

$I(G)$is the minimum cardinally of a maximum independent set of *G*.

Example:

Maximal and minimal independent set therefore $\beta(G) = i(G)$

Definition

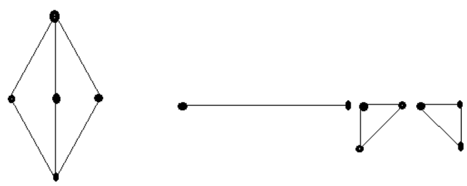
A cut vertex of a graph is one vertex in G whose removal increases the number of components.

Thus, if v is a cut vertex of aconnected graph *G*, then $G - v$ is disconnected.

Definition

A connected graph that has no cut – vertex is called block.

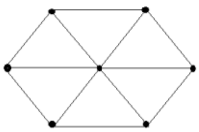
Example



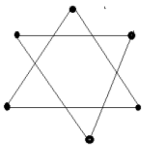
Definition: 1.29

A complement $\bar{G}$ of a graph *G* also has $V(G)$ as its point set. But two points are adjacent in $\bar{G}$ if and only if they are not adjacent in *G*.

Example



$G = K_{1,6}$

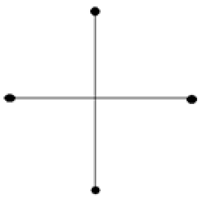


G

Definition

Any complete bipartite graph of the form $k_{1,n}$ is called star graph.

Example:

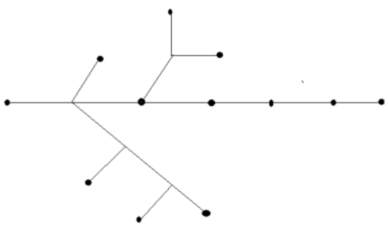


Definition

Let $G = (V, E)$ be a graph, a set $D \subseteq V$ is a dominating set of G is every vertex in $V - D$ is a adjacent to some vertex in D.

A dominating set is minimal if for any $v \in D$, $D - \{V\}$ is not a dominating set of G.

Example



If G is the following tree, than D = $\{v_2, v_{13}, v_6, v_7, v_{10}, v_{15}\}$ is a dominating set.

A graph may have many dominating set of a graph need not have the same cardinality, the minimum cardinality of a dominating set of a graph G is said to be dominating number of G and denoted by γ (G). (ie) γ (G) = {|G| : G is domination set of G}

Definition

The number of vertices in a minimum covering of G is the covering number of G and is denoted by $\beta(G)$.

Definition

Let $G = (V,E)$ and let $S \subset V$ a vertex $v \in S$ is called and enclave of S. if $N[v] \subset S$ and $v \in S$ is called and isolated of S. If $N[v] \subseteq V - S$ a set is said to be enclave less if S has no enclaves.

Definition

A dominating set of a graph G with minimum cardinality is called minimum dominating set and the cardinality of a minimum dominating set is called the dominating number of G is called the dominating by γ (G).

The Cobondage Number of a Graph

Definition

The graphs considered here are finite, undirected without loops and multiple edges having p vertices and q edges. $\lceil X \rceil$is a least integer not less than X. Graphs considered maximum degree at most p-2.

A set D of vertices in a graph $G = (V,E)$ is an dominating set of G if every vertex in V-D is adjacent to some vertex in D. The domination number γ(G) of G is the minimum cardinality of a dominating set. For a survey of results on domination.

The co bondage number b_c (G) of a graph G is the minimum cardinality among the set of edges $X \subseteq$ P2(V)- E2, where P2(V)= $\{X \subseteq V : |X| = 2\}$ such that y(G+X)< γ(G).

Theorem

For any graph G,

1. $b_c(\bar{G}) \leq \delta$ (G)

Where $\bar{G}$and $\delta(\bar{G})$ are the complement and minimum degree of G respectively.

Corollary

For any graph G.

2. $b_c(\bar{G}) \leq p-1\Delta$ (G)

Where Δ (G) is the maximum degree of G.

Now we obtain the exact values of $b_c(G)$ for some standard graphs. Proposition 2. If G = $K_{n1,n2...nt}$, where n1 ≤n2≤ ...≤ nt, then,

3. $b_c(G)=n_1$-l.

Proof

let V = V_{n1}∪ V_{n2}∪ ...∪ Vnt. Then for any two vertices $v \in V_{n1}$ and ω ∈ V_{nj} for 2 ≤aj ≤ n_t {v,w} is a γ -set for G. since each Vni, for 1 ni nt, is independent with |Vni| ≥ 2, by joining each vertex in V_{ni}- {v} to v we obtain a graph which has {v} as a γ set. This proves (3).

Proposition

For any cycle Cp with p 4 vertices,

4. $b_c(C_p)$ = 1, if p = 1 (mod 3);
5. = 2, if p = 2 (mod 3);
6. = 3, other wise

Proof

Let C_p : v_1 v_2… v_p v_1 denote a cycle on p ≥ 4 vertices. We consider the following case.

Case 1

If p=l (mod 3), then by joining by the vertex v_{p-1} to v_1, we obtain a graph G which is a cycle C_{p-1}: v_1 v_2... v_{p-1} v_1 together with a path $v_{p-1}v_pv_1$ This inplies that,

γ (G) = $\gamma(C_{p-1})$

= $\lceil(\mathbf{p}-\mathbf{1})/\mathbf{3}\rceil < \lceil\mathbf{p}/\mathbf{3}\rceil = \gamma(C_p)$

This proves (4).

Case 2

If p=2 (mod 3), then by joining the vertices v_1 and v_p to $C_{p-2}$the resulting graph G is a cycle C_{p-2} :v_1v_2... v_{p-2} v_1 together with a path v_{p-2} $v_{p-1}v_p$ v_1 such that $v_{p-2}$is adjacent to v_p Thus

γ (G) = $\gamma(C_{p-2})$

= $\lceil(\mathbf{p}-\mathbf{2})/\mathbf{3}\rceil < \lceil\mathbf{p}/\mathbf{3}\rceil = \gamma(C_p)$

Hence (5) holds.

Case 3

If p=3 (mod 3), then by adding the edges v_1 v_{p-3}, v_p v_{p-3}, $v_{p-1}v_{p-3}$the resulting graph G is cycle c_{p3} : v_1 v_2... v_{p-3} v_1 together with a path v_{p-3} v_{p-2} $v_{p-1}v_p$ $v_1$such that $v_{p-3}$is adjacent to both $v_{p-1}$and v_p. Hence,

γ (G) = $\gamma(C_{p-2})$

= $\lceil(\mathbf{p}-\mathbf{3})/\mathbf{3}\rceil < \lceil\mathbf{p}/\mathbf{3}\rceil = \gamma(C_p)$

thus (6) holds.

Proposition

For any path P_p with p ≥4 vertices.

7. $b_c(P_P)$ = 1, if p≡l (mod 3);
8. = 2, if p≡2 (mod 3);
9. = 3, if p ≡ 3 (mod 3).

Proof

Proofs (7), (8) and (9) are similar to that of proofs of (4),(5) and (6), respectively.

Theorem

Let T be a tree with at least two cutvertices such that each cutvertex is adjacent to an end vertex. Then,

10. $b_c(T)$ = r

where r is the minimum number of end vertices adjacent to a cut vertex.

Proof

Let S be the set of all cut vertices of T. Then S is a γ set for T. Let u∈S be a cutvertex which is adjacent to minimum number of end vertices u_1, u_2, ….,u_r to v the graph obtained has S-{u} as a γ - set. This proves (10).

Now we obtain some more upper bounds on $b_c(G)$.

Theorem

For any graph G.

11. $b_c(G) \leq \Delta(G)$+l

Furthermore, the bound is attained if and only if every γ - set D of G satisfying the following conditions.

D is independent

every vertex in D is of maximum degree

every vertex in V-D is adjacent to exactly one vertex in D.

Proof:

Let D be a γ - set of G. We consider the following cases.

Case 1

Suppose D is not independent. Then there exist two adjacent vertices, u, $v \in D$ Let $S \subset V\text{-}D$. Such that for each vertex $\omega \in S$, $N(u_1)D=\{v\}$. Then by joining each vertex in S to u, we see that $D\text{-}\{v\}$ is a γ - set of the resulting graph.

Thus,

12. $b_c(G) \leq |S| \leq \Delta(G)\text{-}l$

Case 2:

Suppose D is independent. Then each vertex $v \in D$ is an isolated vertex in < D >. Let S be a set defined in Case 1. Since D has at least *two* vertices, by joining each vertex in S $\cup\{v\}$ to some vertex $\omega \in D\{v\}$, we obtain a graph which has D-$\{v\}$ as a γ - set Hence.

13. $b_c(G) \leq |S \cup \{v\}| \leq \Delta(G)+l$

The second part of the theorem directly follows from Cases 1 and 2.

Corollary

For any graph G,

14. $b_c(G) \leq \min\{p\text{-}A(G)\text{ -}1, \Delta(G)+l)$

Theorem

For any graph G.

11. $b_c(G) \leq p\text{-}l$

Further, the bound is attained if and only if G= $\bar{K}2$.

Proof

Since $\Delta(G) \leq p\text{-}2$, (13), follow from (11).

Suppose the bound is attained. Then by (1), it follows that $\bar{G}$ Kp. Suppose G has at least three vertices. Then $b_c(G) = 1 < p\text{-}1$, a contradiction. This implies that $\bar{G} = K_2$ and hence G —$\bar{K}_2$ Converse is immediate.

The next result improve the inequality (13).

Theorem

For any graph G with $p \geq 3$ vertices.

15. $b_c(G) \leq p\text{-}2$

Further, the bound is attained if and only if $G=2K_2$ or $\bar{K}_3$ or $K_2 \cup K_1$.

Proof

Suppose the bound is attained. Then $\Delta(G)= 1$. Suppose $p \geq 5$. Then, $b_c(G) \leq p\text{-}3$, a contradiction. This implies that p = 3 or 4. Forp = 3, obviously G = $\bar{K}_3$ or$\mathbf{K_2} \cup \mathbf{K_1}$. If p = 4 and G contains an isolate, then, $b_c(G) = 1$, a contradiction. This proves that G = $\mathbf{2K_2}$. Converse is easy to prove.

The bondage number b(G) of G is the minimum cardinality among the sets of edges $X \subseteq E$ such that $\gamma(G\text{-}X) > \gamma(G)$.

Theorem A (2), For any nontrivial tree T, $b(T) \leq 2$.

As a consequence of Theorem 6 and Theorem A, we have.

Theorem

Let T be a tree with diam (T) = 5 and has exactly two cut vertices which are adjacent to end vertices and further they have same degree. Then,

16. $b_c(T) \geq b(T)+l$

where diam (T) is the diameter of T.

Theorem

For any tree T,

$b_c(T) \leq 1+\min\{\deg u)$

where u is a cut vertex adjacent to an end vertex.

Proof

Since there exists a γ -set containing u, by applying same technique as we used in proving (11) we get (16).

The next result relates to $b_c(G)$ and $b_c(T)$.

Theorem

Let T be a spanning tree of G such that $\gamma(T) = \gamma(G)$. Then

17. $bc(G) \leq bc(T)$.

Proof

Let X be a b_c-set of T. Then exists a set $X' \subseteq X$ such that $\gamma(G\text{-}FX') < \gamma(G)$. This proves (17).

Now we obtain a relationship between $b_c(G)$ and y(G).

Theorem

For any graph G,

18. $b_c(G) + \gamma(G) \leq p+l$.

Further, the equality holds if and only if $G = \overline{K}_P$

Proof

Let D be a γ -set of G. Let $v \in V-D$. Then there exists a vertex u∈D such that v is adjacent to it. Since there exists a vertex w∈D-{u}.by joining the vertices of ((V-D)-{v}∪{ ω)} to u, we see that D-{co} is a γ -set of the result of graph. This proves (18). Now we prove the second part.

Suppose the equality holds. On the contrary, G $\neq \overline{K}_P$ Then by above. $b_c(G) \leq p-\gamma(G)$, a contradiction. This proves that $G = \overline{K}_P$ Converse is obvious.

The next result sharpnes the inequality (18).

Theorem

Let D be a γ- set of G. If there exists a vertex $v \in D$ which is adjacent to every other vertex in D, then,

19. $b_c(G) \leq p-\gamma(G)-l$.

Proof

This follows from (2), since $\Delta(G) \geq \deg v \geq \gamma(G)$.

Lastly we obtain a Nordhaus - Gaddum type result.

Theorem

Let G be a graph with $p \geq 4$ vertices such that neither $\bar{G}$ nor g is $2K_2$. Then,

20. $b_c(G)+b_c(\bar{\boldsymbol{G}}) \leq 2(p-3)$.

The equality holds if and only if G = p4 or C5.

Proof

Follows from Theorem 8.

Suppose the equality holds. Then, $\Delta(G), \Delta(\bar{G}) \leq 2$.

Suppose $\Delta(G)$ or $\Delta(\bar{G})$-1. say A (G) = 1. Then, $\Delta(\bar{G})- \geq 3$, a contradiction. Hence, $\Delta(G) = \Delta(\bar{G}) = 2$. If $p \geq 6$, then $\Delta(\bar{G}) \geq 3$, a contradiction. Thus, p = 4 or 5. This implies that $G = p_4$ or C_5. Converse is easy to prove.

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